TWISTED VERTEX REPRESENTATIONS VIA SPIN GROUPS AND THE MCKAY CORRESPONDENCE

IGOR B. FRENKEL, NAIHUAN JING, AND WEIQIANG WANG

ABSTRACT. We establish a twisted analog of our recent work on vertex representations and the McKay correspondence. For each finite group Γ and a virtual character of Γ we construct twisted vertex operators on the Fock space spanned by the super spin characters of the spin wreath products $\Gamma \wr \widetilde{S}_n$ of Γ and a double cover of the symmetric group S_n for all n. When Γ is a subgroup of $SL_2(\mathbb{C})$ with the McKay virtual character, our construction gives a group theoretic realization of the basic representations of the twisted affine and twisted toroidal algebras. When Γ is an arbitrary finite group and the virtual character is trivial, our vertex operator construction yields the spin character tables for $\Gamma \wr \widetilde{S}_n$.

1. INTRODUCTION

The connection among the direct sum of Grothendieck groups of the symmetric groups S_n for all n and the theory of symmetric functions [M, Z] has a simple interpretation in terms of a Heisenberg algebra and vertex operators ([F1], see part one of [J1]). In the recent works [W, FJW1] we have realized a generalization of such a connection by substituting the symmetric group S_n with the wreath product $\Gamma_n = \Gamma \wr S_n$ associated to an arbitrary finite group Γ. Moreover, we introduced a crucial modification of this connection that, in the case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, yields a group theoretic realization of the affine Lie algebra $\widehat{\mathfrak{g}}$ [FK, Se] and of the toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}$ [F2, MRY], where $\mathfrak{g}$ is a complex simple Lie algebra of ADE type whose Dynkin diagram is related to Γ via the McKay correspondence [Mc].

The main goal of the present work is to extend the above results to realize the twisted basic representation of an affine Lie algebra $\widehat{\mathfrak{g}}[-1]$ and its toroidal counterpart by means of a spin cover $\widetilde{\Gamma}_n$ of the wreath product Γ_n associated to a subgroup Γ of $SL_2(\mathbb{C})$.

1991 *Mathematics Subject Classification.* Primary: 17B, 20.

Key words and phrases. twisted vertex operators, double cover of wreath product, spin characters.

Research of Frenkel is supported by NSF grant DMS-9700765; research of Jing is supported by NSF grant DMS-9970493.

The twisting of the basic representation of the affine Lie algebra under consideration is determined by the multiplication by -1 on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and can be viewed as an odd counterpart of the even (untwisted) case. This twisting was originally introduced as the first step towards the construction of the Moonshine module for the Monster group in [FLM1, FLM2]. As in the homogeneous case one starts with a representation of the Heisenberg subalgebra $\widehat{\mathfrak{h}}[-1]$ and reconstructs the rest of the twisted affine Lie algebra $\widehat{\mathfrak{g}}[-1]$ using the twisted vertex operators.

The representation theory of the spin group $\widetilde{S}_n$ which is a double cover of the symmetric group S_n was initiated by I. Schur [S] (also see [Jo] for an exposition). Its connection with vertex operators was further studied in [J1]. These results will play an important role in our present work. The representation theory of $\widetilde{\Gamma}_n$ was also studied in [HH] from a Hopf algebra viewpoint.

In order to work effectively only with the spin representations of $\widetilde{\Gamma}_n$, i.e, those which do not factor through Γ_n, we adopt the approach of [Jo] by introducing a superalgebra structure on the group algebra of $\widetilde{\Gamma}_n$ and consider its supermodules. It turns out that the superstructure is preserved under the main operations such as induction and restriction. The direct sum of the Grothendieck groups of spin supermodules of $\widetilde{\Gamma}_n$ carries a natural Hopf algebra and we remark that a Hopf algebra was constructed in [HH] on a different space. This allows us to realize the vertex operators acting in the twisted vertex representations constructed from the sum of the Grothendieck rings. Our group theoretic method naturally recovers the basic representations of twisted affine Lie algebras $\widehat{\mathfrak{g}}[-1]$ [LW, FLM1, FLM2]. As in [FJW1] we realize this by introducing a modified bilinear form associated to the McKay virtual character ξ which is twice the trivial character minus the character of the two-dimensional natural representation of Γ in $SL_2(\mathbb{C})$.

Much of our construction is valid for an arbitrary finite group Γ and we have introduced the modified bilinear form associated to an arbitrary virtual character ξ of Γ as well. In the special case when ξ is the trivial character the twisted vertex operators generate an infinite dimensional generalized Clifford algebra, which recovers the twisted boson-fermion correspondence. We further obtain the super character tables of the spin group $\widetilde{\Gamma}_n$ for all n, generalizing the results of [J1].

One may generalize the results of this paper to the quantum case as it was done in [FJW2] for the homogeneous picture of quantum affine algebras [FJ]. Our results also suggest that various previous constructions associated to (quantum) vertex representations admit remarkable

interpretation via Grothendieck rings of certain finite groups which are variations of wreath products, though every new step in this direction is unpredictable and brings new surprises. It is a very interesting and challenging problem to find such a group theoretic realization.

The organization of the paper is as follows. In Sect. 2 we present the representation theory and structures of the spin group $\widetilde{\Gamma}_n$. In Sect. 3 we review superalgebras and supermodules and define the Hopf algebra of the super spin characters of $\widetilde{\Gamma}_n$. In Sect. 4 we introduce the weighted bilinear forms in the Grothendieck rings of supermodules and construct basic spin supermodules. In Sect. 5 we define the twisted Heisenberg algebras and their Fock spaces. In Sect. 6 we establish the isometry between the sum of Grothendieck rings of supermodules of $\widetilde{\Gamma}_n$ and the Fock space of a twisted Heisenberg algebra. In Sect. 7 we construct twisted vertex operators via the induction and restriction functors on the Grothendieck rings. In Sect. 8 we obtain the twisted basic representation of the affine Lie algebras $\widehat{\mathfrak{g}}[-1]$ and the corresponding toroidal algebras. In Sect. 9 we derive the super spin character tables of $\widetilde{\Gamma}_n$ for all n from the twisted boson-fermion correspondence.

2. A double cover of the wreath product

2.1. **The spin group $\widetilde{S}_n$.** In this subsection we discuss some of the basic properties of the double covers of the symmetric group, which were introduced by Schur in his seminal paper [S]. We will adopt the modern account [Jo] of Schur's theory.

Let S_n be the symmetric group of n letters, and we use the convention of multiplying permutations from right to left (different from [S, Jo]). The spin group $\widetilde{S}_n$ is the finite group generated by z and $t_i, i = 1, \cdots, n-1$ subject to the relations:

$$z^2 = 1, \quad t_i^2 = (t_i t_{i+1})^3 = z, \tag{2.1}$$

$$t_i t_j = z t_j t_i, \qquad i > j+1, \tag{2.2}$$

$$z t_i = t_i z. \tag{2.3}$$

Let θ_n be the homomorphism from $\widetilde{S}_n$ to S_n sending t_i to the transposition $(i, i+1)$ and z to 1. We see that $\widetilde{S}_n$ is a central extension of S_n by the cyclic group $\mathbb{Z}_2$:

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \widetilde{S}_n \xrightarrow{\theta_n} S_n \longrightarrow 1,$$

where the embedding ι sends the order 2 element in $\mathbb{Z}_2$ to z. Schur [S] determined that $H^2(S_n, \mathbb{C}^*) \simeq \mathbb{Z}_2$ for $n > 3$. The group $\widetilde{S}_n$ is one of the two double covers of the symmetric group S_n ($n > 3$). Our results

in this paper can be easily translated to the other double cover (cf. [S, J1]).

The group $\widetilde{S}_n$ has a parity given as follows. Let d be the homomorphism from the free group generated by $\{t_i, z\}(i = 1, \cdots, n-1)$ to $\mathbb{Z}_2$ by $d(t_i) = 1$, $i = 1, \cdots, n-1$ and $d(z) = 0$. It is easily seen that d preserves the relations (2.1-2.3). Thus it defines a homomorphism from $\widetilde{S}_n$ to $\mathbb{Z}_2$, which we still denote by d. An element $x \in \widetilde{S}_n$ is called *even* (resp. *odd*) if $d(x) = 0$ (resp. $d(x) = 1$). The parity in $\widetilde{S}_n$ given by d lifts the standard notion of even and odd permutations in the symmetric group S_n.

The spin group $\widetilde{S}_n$ has a cycle presentation due to J. H. Conway and others (see [Ws]). Embed $\widetilde{S}_n$ into $\widetilde{S}_{n+1}$ by identifying their first $n-1$ generators $t_i, i = 1, \cdots, n-1$. For $i = 1, \cdots, n$ we define $x_i = t_i t_{i+1} \cdots t_n \cdots t_{i+1} t_i \in \widetilde{S}_{n+1}$. For a sequence $i_1, \cdots, i_m$ of distinct integers from $\{1, 2, \cdots, n\}$ we can define cycles in $\widetilde{S}_n$ as follows.

$$[i_1 i_2 \cdots i_m] = \begin{cases} z, & m = 1, \\ x_{i_1} x_{i_m} x_{i_{m-1}} \cdots x_{i_1}, & 1 < m \leq n. \end{cases} \tag{2.4}$$

It is known that $\theta_n([i_1 i_2 \cdots i_m]) = (i_1 i_2 \cdots i_m)$ and $\theta_{n+1}(x_i) = (i, n+1)$. We list some useful identities for the cycles.

$$x_j [i_1 i_2 \cdots i_m] = z^{m-1} [i_1 i_2 \cdots i_m] x_j, \quad (j \neq i_s), \quad x_j^2 = z, \tag{2.5}$$

$$[i_1 i_2 \cdots i_m]^{-1} = [i_m \cdots i_2 i_1], \tag{2.6}$$

$$[i_1 i_2 \cdots i_m] = z^{m-1} [i_2 i_3 \cdots i_m i_1], \tag{2.7}$$

$$[i_1 i_2 \cdots i_m][j_1 j_2 \cdots j_k] = z^{(m-1)(k-1)} [j_1 j_2 \cdots j_k][i_1 i_2 \cdots i_m], \tag{2.8}$$

$$[i, i+1, \cdots, i+j-1] = z^{j-1} t_i t_{i+1} \cdots t_{i+j-2}, \tag{2.9}$$

where the cycles $[i_1 i_2 \cdots i_m]$ and $[j_1 j_2 \cdots j_k]$ are disjoint.

Proposition 2.1. [Jo] *Each element of $\widetilde{S}_n$ can be presented as*

$$z^p [i_1 i_2 \cdots i_m][j_1 j_2 \cdots j_k] \cdots,$$

where $\{i_1 \cdots i_m\}, \{j_1 \cdots j_k\}, \cdots$ is a partition of the set $\{1, 2, \cdots, n\}$ and $p = 0, 1$. If $z^p c_1 c_2 \cdots c_l = z^{p'} c'_1 c'_2 \cdots c'_{l'}$ are two expressions of the same element in terms of cycles c_i and c'_i, then $l = l'$ and there is a permutation $\sigma \in S_l$ such that

$$c'_i = c_{\sigma(i)} z^{m_i}, \qquad m_i \equiv |c_i| - 1 (mod\, 2),$$

where $|c_i|$ denotes the length of the cycle c_i. Moreover if $[i_1 \cdots i_k] = z^m [j_1 \cdots j_k]$, then $j_s = \sigma(i_s)$ for a cyclic permutation σ of $\{i_1, \cdots, i_k\}$.

Let λ be a partition and we identify λ with its Young diagram consisting of l rows of $\lambda_1, \cdots, \lambda_l$ squares respectively aligned to the left. A *tableau* T_λ of shape λ is a numbering of the squares with integers $1, 2, \cdots, |\lambda|$, each appearing exactly once. For each tableau T_λ of shape λ with a numbering $a_{11}, \cdots a_{1\lambda_1}, a_{21}, \cdots, a_{2\lambda_2}, \cdots, a_{l1}, \cdots, a_{l\lambda_l}$ we define the element t_λ of $\widetilde{S}_n$ to be

$$t_\lambda = [a_{11} \cdots a_{1\lambda_1}][a_{21} \cdots a_{2\lambda_2}] \cdots [a_{l1} \cdots a_{l\lambda_l}]. \tag{2.10}$$

The permutation $\prod_{i=1}^{l}(a_{i1} \cdots a_{i\lambda_i})$ associated with t_λ will be denoted by $s(\lambda)$. It follows from Proposition 2.1 that the general element in $\widetilde{S}_n$ is of the form $z^p t_\lambda$. For a permutation $s \in S_n$ we also define $t_\lambda^s = \prod_{i=1}^{l}[s(a_{i1}) \cdots s(a_{i\lambda_i})]$.

The following can be checked by induction using (2.5) and (2.8).

Lemma 2.2. *For any two elements t_λ, t_μ in $\widetilde{S}_n$ associated to tableaux T_λ and T_μ we have that*

$$t_\mu t_\lambda t_\mu^{-1} = z^{d(\lambda)d(\mu)} t_\lambda^{s(\mu)}.$$

2.2. **The spin wreath product $\widetilde{\Gamma}_n$.** In this subsection we introduce the main finite group $\widetilde{\Gamma}_n$ in this work, and extend our discussion from $\widetilde{S}_n$ to $\widetilde{\Gamma}_n$.

Let Γ be a finite group with $r+1$ conjugacy classes. We denote by $\Gamma^* = \{\gamma_i\}_{i=0}^r$ the set of complex irreducible characters, where γ_0 stands for the trivial character, and by Γ_* the set of conjugacy classes. The character value $\gamma(c)$ of $\gamma \in \Gamma^*$ at a conjugacy class $c \in \Gamma_*$ yields the character table $\{\gamma(c)\}$ of Γ.

Let $R(\Gamma) = \oplus_{i=0}^r \mathbb{C}\gamma_i$ be the space of complex valued class functions on Γ. For $c \in \Gamma_*$ let ζ_c be the order of the centralizer of an element in the class c, so the order of the class is then $|\Gamma|/\zeta_c$. The usual bilinear form on $R(\Gamma)$ is defined as follows:

$$\langle f, g \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x) g(x^{-1}) = \sum_{c \in \Gamma_*} \zeta_c^{-1} f(c) g(c^{-1}),$$

where c^{-1} denotes the conjugacy class $\{x^{-1}, x \in c\}$. Clearly $\zeta_c = \zeta_{c^{-1}}$. We will often write $\langle\ ,\ \rangle$ for $\langle\ ,\ \rangle_\Gamma$ when no ambiguity may arise. It is well known that

$$\begin{aligned} \langle \gamma_i, \gamma_j \rangle &= \delta_{ij}, \\ \sum_{\gamma \in \Gamma^*} \gamma(c') \gamma(c^{-1}) &= \delta_{c,c'} \zeta_c, \quad c, c' \in \Gamma_*. \end{aligned} \tag{2.11}$$

Thus $R_{\mathbb{Z}}(\Gamma) = \oplus_{i=0}^{r} \mathbb{Z}\gamma_i$ endowed with this bilinear form becomes an integral lattice in $R(\Gamma)$.

Given a positive integer n, let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the n-th direct product of Γ, and let Γ^0 be the trivial group. The spin group $\widetilde{S}_n$ acts on Γ^n through the action of the group S_n by permuting the indices: $t_\lambda(g_1, \cdots, g_n) = (g_{s(\lambda)^{-1}(1)}, \cdots, g_{s(\lambda)^{-1}(n)})$, and $z(g_1, \cdots, g_n) = (g_1, \cdots, g_n)$. The wreath product $\widetilde{\Gamma}_n = \Gamma \wr \widetilde{S}_n$ of Γ with $\widetilde{S}_n$ is defined to be the semi-direct product

$$\widetilde{\Gamma}_n = \Gamma^n \rtimes \widetilde{S}_n = \{(g,t) | g = (g_1, \cdots, g_n) \in \Gamma^n, t \in \widetilde{S}_n\}$$

with the multiplication

$$(g,t) \cdot (h,s) = (g\, t(h), ts).$$

Note that $\widetilde{\Gamma}_n$ reduces to $\widetilde{S}_n$ when Γ is trivial. Clearly $\widetilde{\Gamma}_n$ is a central extension of Γ_n by $\mathbb{Z}_2$ and $|\widetilde{\Gamma}_n| = 2n!|\Gamma|^n$.

We define a a parity for $\widetilde{\Gamma}_n$ by extending the parity of $\widetilde{S}_n$. Let $d : \widetilde{\Gamma}_n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ be the homomorphism from $\widetilde{\Gamma}_n$ to $\mathbb{Z}_2$ given by

$$d(g, t_i) = 1, \qquad d(g, z) = 0. \tag{2.12}$$

Clearly the degree 0 subset $\widetilde{\Gamma}_n^0$ is the wreath product $\Gamma \wr \widetilde{A}_n$, where A_n is the alternating group, and the degree one part $\widetilde{\Gamma}_n^1$ is the complementary subset.

Let τ be a section from Γ_n to $\widetilde{\Gamma}_n$ such that $\theta\tau = 1$. An element $x \in \Gamma_n$ is called *split* if $\tau(x)$ is not conjugate to $z\tau(x)$. Otherwise x is said to be *non-split*. Clearly this definition does not depend on the choice of the section τ and two conjugate elements are simultaneously split or non-split. A conjugacy class of Γ_n is called *split* if its elements are split. We will also say that an element $x \in \widetilde{\Gamma}_n$ is split (resp. non-split) if $\theta(x)$ is split (resp. non-split). Clearly the class C_ρ splits if and only if the preimage $\theta_n^{-1}(C_\rho)$ splits into two conjugacy classes in $\widetilde{\Gamma}_n$.

Any representation π of Γ_n can be viewed as a representation of $\widetilde{\Gamma}_n$. Such a representation π of $\widetilde{\Gamma}_n$ satisfies the property $\pi(z) = Id$. A representation π of $\widetilde{\Gamma}_n$ is called *spin* if $\pi(z) = -Id$. It follows that the characters of spin representations vanish on non-split classes. In this paper we only consider spin representations.

We remark that spin representations are sometimes referred as negative or projective representations in the literature.

2.3. **Conjugacy classes of $\widetilde{\Gamma}_n$.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be a partition of the integer $|\lambda| = \lambda_1 + \cdots + \lambda_l$, where $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. The integer l is called the *length* of the partition λ and is denoted by $l(\lambda)$. We will

identify the partition $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ with $(\lambda_1, \lambda_2, \cdots, \lambda_l, 0, \cdots, 0)$. We will also make use of another notation for partitions:

$$\lambda = (1^{m_1}2^{m_2}\cdots),$$

where m_i is the number of parts in λ equal to i. The number $n(\lambda')$ is defined to be $\sum_i \binom{\lambda_i}{2}$, where λ' is the dual partition associated to λ. We will use the dominance order on partitions. For two partitions λ and μ we write $\lambda \geqslant \mu$ if $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$, etc.

A partition λ is called *strict* if its parts are distinct integers (excluding the trivial parts of zero), in which case all the multiplicities m_i are 1.

We will use partitions indexed by Γ_* and Γ^*. For a finite set X and $\rho = (\rho(x))_{x\in X}$ a family of partitions indexed by X, we write

$$\|\rho\| = \sum_{x\in X} |\rho(x)|.$$

It is convenient to regard $\rho = (\rho(x))_{x\in X}$ as a partition-valued function on X. We denote by $\mathcal{P}(X)$ the set of all partitions indexed by X and by $\mathcal{P}_n(X)$ the set of all partitions in $\mathcal{P}(X)$ such that $\|\rho\| = n$. The total number of parts, denoted by $l(\rho) = \sum_x l(\rho(x))$, in the partition-valued function $\rho = (\rho(x))_{x\in X}$ is called the length of ρ. The *dominance order* is extended to partition-valued functions as follows. We define $\rho \geqslant \pi$ if $\rho(x) \geqslant \pi(x)$ for each x. We say that $\rho \gg \pi$ if $\rho(x) \geqslant \pi(x)$ and $\rho(x) \neq \pi(x)$ for *each* $x \in X$. For a partition-valued function ρ we define

$$n(\rho') = \sum_c n(\rho(c)') = \sum_{c,i} \binom{\rho_i(c)}{2}. \tag{2.13}$$

Let $\mathcal{OP}(X)$ be the set of partition-valued functions $(\rho(x))_{x\in X}$ in $\mathcal{P}(X)$ such that all parts of the partitions $\rho(x)$ are odd integers, and let $\mathcal{SP}(X)$ be the set of partition-valued functions $\rho : X \longrightarrow \mathcal{P}$ such that each partition $\rho(x)$ is strict. When X consists of a single element, we will omit X and simply write $\mathcal{P}$ for $\mathcal{P}(X)$, thus the notation $\mathcal{OP}$ or $\mathcal{SP}$ will be used similarly.

Lemma 2.3. $|\mathcal{OP}_n(X)| = |\mathcal{SP}_n(X)|$.

Proof. The generating function of the cardinalities of strict partition-valued functions is

$$\prod_{x\in X}\prod_{n=1}^{\infty} \frac{1}{1-q_x^{2n-1}} = \prod_{x\in X}\prod_{n=1}^{\infty} \frac{1-q_x^{2n}}{(1-q_x^{2n-1})(1-q_x^{2n})} = \prod_{x\in X}\prod_{n=1}^{\infty}(1+q_x^n),$$

which is the generating function of $\mathcal{SP}_n(X)$. ∎

We also define a parity on partitions. For each partition λ we define $d(\lambda) = |\lambda| - l(\lambda)$. For a partition-valued function $\rho = (\rho(x))_{x\in X}$ we define $d(\rho) = \sum_x |\rho(x)| = \|\rho\| - l(\rho)$. It is clear that the conjugacy class of type λ in S_n is even if and only if $d(\lambda)$ is even. We define the parity of the partition-valued function ρ to be the parity of $d(\rho)$. We define

$$\mathcal{P}_n^0(X) = \{\lambda \in \mathcal{P}_n(X) | \quad d(\rho) \equiv 0 (mod\, 2)\}, \tag{2.14}$$

$$\mathcal{P}_n^1(X) = \{\lambda \in \mathcal{P}_n(X) | \quad d(\rho) \equiv 1 (mod\, 2)\}, \tag{2.15}$$

and define $\mathcal{SP}_n^i(X) = \mathcal{P}_n^i(X) \cap \mathcal{SP}_n(X)$ for $i = 0, 1$.

We now recall the description of conjugacy classes of Γ_n [M]. Let $x = (g, \sigma)$ be an element in a conjugacy class of Γ_n, where $g = (g_1, \cdots, g_n)$. For each cycle $y = (i_1 i_2 \cdots i_k)$ in the permutation σ the element $g_y = g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$ is called the *cycle-product* of x corresponding to the cycle y. For each $c \in \Gamma_*$ and $i \geq 0$ let $m_i(c)$ be the number of i-cycles in the permutation σ such that the cycle products g_y lie in the conjugacy class c. Then $c \to \rho(c) = (1^{m_1(c)} 2^{m_2(c)} \cdots)$ defines a partition-valued function on Γ_*. It is known that the partition-valued function $(\rho(c))_{c\in\Gamma_*}$ is in one-to-one correspondence to the conjugacy class of $x = (g, \sigma)$ in Γ_n and is called the *type* of the conjugacy class. We will also say that an element has conjugacy type ρ if this element is contained in the conjugacy class.

Let $(-1)^d$ be the representation of $\widetilde{\Gamma}_n$ given by $x \longmapsto (-1)^{d(x)}$. A representation π of $\widetilde{\Gamma}_n$ is called a *double spin* representation if

$$(-1)^d \pi \simeq \pi.$$

If $\pi' = (-1)^d \pi \neq \pi$, then π' and π are called *associate spin* representations of $\widetilde{\Gamma}_n$.

The following result was proved in [J1] for a double cover of any finite group.

Proposition 2.4. *The number of split conjugacy classes of* Γ_n *is equal to the number of irreducible spin representations of* $\widetilde{\Gamma}_n$.

2.4. **Split conjugacy classes of $\widetilde{\Gamma}_n$.** We fix an order of conjugacy classes of Γ: $c^0 = \{1\}, c^1, \cdots, c^r$. For each partition-valued function $\rho = (\rho(c)) \in \mathcal{P}_n(\Gamma_*)$, we let $t_{\rho(c^i)}$ be the element of $\widetilde{S}_n$ associated to a tableau $T_{\rho(c^i)}$ of shape $\rho(c^i)$ using the numbers $\sum_{j\leq i-1} |\rho(c^j)| + 1, \cdots, \sum_{j\leq i} |\rho(c^j)|$ and we define the element t_ρ to be

$$t_\rho = t_{\rho(c^0)} t_{\rho(c^1)} \cdots t_{\rho(c^r)}, \tag{2.16}$$

which depends on the sequence T_ρ of the tableaux $T_{\rho(c^0)}, \cdots, T_{\rho(c^r)}$. We remark that the general element of $\widetilde{\Gamma}_n$ is of the form $(g, z^p t_\rho)$, where ρ is the type of the conjugacy class of $(g, z^p t_\rho)$.

The following theorem is well-known in the case of $\Gamma = \{1\}$ (cf. [Jo] and [St]).

Theorem 2.5. *Let $\rho = (\rho(c))_{c\in\Gamma_*}$ be the type of a conjugacy class C_ρ in Γ_n. Then the preimage $\theta_n^{-1}(C_\rho)$ splits into two conjugacy classes in $\widetilde{\Gamma}_n$ if and only if*

(1) when the class C_ρ is even and all the $\rho(c)$ $(c \in \Gamma_*)$ are partitions with odd integer parts, i.e., $\rho \in \mathcal{OP}_n(\Gamma_*)$;

(2) when the class C_ρ is odd and all the $\rho(c)$ $(c \in \Gamma_*)$ are strict partitions, i.e., $\rho \in \mathcal{SP}_n^1(\Gamma_*)$.

Proof. (1) Let $d(\rho)$ be even and let each partition $\rho(c)$ have odd integer parts. Assume on the contrary that (g, t_ρ) and $z(g, t_\rho)$ are conjugate in $\theta_n^{-1}(C_\rho)$, where t_ρ is associated to a sequence of tableaux (see Eqn. (2.16)). Then for some $(h, t_\mu) \in \widetilde{\Gamma}_n$

$$
\begin{aligned}
&(h, t_\mu)(g, t_\rho)(h, t_\mu)^{-1} \\
=&(h \cdot s(\mu)(g) \cdot s(\mu)s(\rho)s(\mu)^{-1}(h^{-1}), t_\mu t_\rho t_\mu^{-1}) \\
=&(h \cdot s(\mu)(g) \cdot s(\rho)(h^{-1}), t_\mu t_\rho t_\mu^{-1}) = (g, z t_\rho),
\end{aligned}
\tag{2.17}
$$

where we have used the fact that $s(\rho)s(\mu) = s(\mu)s(\rho)$. It follows from Lemma 2.2 that $zt_\rho = z^{d(\rho)d(\mu)} t_\rho^{s(\mu)} = t_\rho^{s(\mu)}$, since $d(\rho) = 0\,(mod\,2)$. Let $t_\rho = c_1 c_2 \cdots c_l$ and $t_\rho^{s(\mu)} = c_1' c_2' \cdots c_l'$ be their cycle representations. Then $c_i = z^{m_i} c'_{\nu(i)}$ and $m_i = |c_i| - 1(mod\,2)$ for some $\nu \in S_l$ by Proposition 2.1. Since each cycle length $|c_i|$ is odd, all the cycles mutually commute with each other. Substituting $c_i = z^{m_i} c'_{\nu(i)}$ back and rearranging the cycles, we have

$$1 = z^{1+\sum_i (|c_i|-1)} = z^{1+d(\rho)} = z,$$

which is a contradiction.

Now suppose that for some $c \in \Gamma_*$ there is an even cycle in $\rho(c)$ of the class C_ρ of type ρ. That is, there is an element $(g, t_\rho) \in \theta^{-1}(C_\rho)$ such that $t_\rho = \cdots [i_1 i_2 \cdots i_{2k}] \cdots$. Consider the element $(h, t_\mu) \in \widetilde{\Gamma}_n$, where $t_\mu = [i_1 i_2 \cdots i_{2k}]$ and $h = (h_1, \cdots, h_n)$ with $h_j = 1$ for $j \neq i_s$ and $h_{i_s} = g_{i_s}, s = 1, \cdots, 2k$. We claim that

$$(h, t_\mu)(g, t_\rho)(h, t_\mu)^{-1} = (hs(\mu)(g)s(\rho)(h^{-1}), t_\mu t_\rho t_\mu^{-1}) = (g, zt_\rho),$$

which is shown by two steps. First we consider the jth component of $hs(\mu)(g)s(\rho)(h^{-1})$ in Γ^n. It equals $1 \cdot g_j \cdot 1 = g_j$ when $j \neq i_s$, and it

equals $g_{i_s}g_{i_{s-1}}g_{i_{s-1}}^{-1} = g_{i_s}$ for $j = i_s$. Secondly we have

$$t_\mu t_\rho t_\mu^{-1} = \cdots [i_2 i_3 \cdots i_{2k} i_1] \cdots = z t_\rho$$

by using Eqn. (2.7) and $d(\rho) \equiv 0\,(mod\,2)$ again. Thus (g, t_ρ) is conjugate to $z(g, t_\rho)$. Therefore all partitions $\rho(c)$ must be from $\mathcal{OP}(\Gamma_*)$ if $\theta^{-1}(C_\rho)$ splits.

(2) Let $d(\rho)$ be odd. Assume all partitions $\rho(c)$ are strict partitions. If on the contrary (g, t_ρ) is conjugate to $z(g, t_\rho)$, then using $d(\rho) = 1$ we have as in (2.17) that $zt_\rho = z^{d(\mu)} t_\rho^{s(\mu)}$ for the permutation $s(\mu) \in S_n$ associated to some $\mu \in \mathcal{P}_n(\Gamma_*)$. Let $t_\rho = c_1 c_2 \cdots c_l$ and $t_\rho^{s(\mu)} = c_1' c_2' \cdots c_l'$ be their cycle representations. Then $c_i = z^{|c_i|-1} c'_{\nu(i)}$ for some $\nu \in S_l$. Since $s(\mu)$ cyclically permutes the indices in each cycles of $s(\rho)$ we have $d(\mu) = d(\rho)$. On the other hand, note that each cycle c_i corresponds to one part in $\rho(c)$ for some $c \in \Gamma_*$ and any conjugation of c_i still corresponds to a part in the same $\rho(c)$. When we plug the equations $c_i = z^{|c_i|-1} c'_{\nu(i)}$ back to $zt_\rho = z^{d(\mu)} t_\rho^{s(\mu)}$ we see that ν is actually the identity since $\rho(c)$ is strict. Therefore $z^{1+\sum_i(|c_i|-1)} = z^{d(\mu)}$. Then $d(\rho) = \sum_i(|c_i|-1) \equiv d(\mu)+1\,(mod\,2)$, which is a contradiction. Hence $\theta_n^{-1}(C_\rho)$ splits.

Now suppose $\theta_n^{-1}(C_\rho)$ splits. If there are two identical parts in $\rho(c)$ for some conjugacy class $c \in \Gamma_*$, say $t_\rho = \cdots [i_1 \cdots i_k][j_1 \cdots j_k] \cdots$ for $(g, t_\rho) \in \widetilde{\Gamma}_n$. Then the cycle-products of these two identical parts are conjugate, i.e., there exists an element $x \in \Gamma$ such that

$$x g_{j_k} g_{j_{k-1}} \cdots g_{j_1} x^{-1} = g_{i_k} g_{i_{k-1}} \cdots g_{i_1}. \tag{2.18}$$

Consider the element (h, t_μ) such that $t_\mu = [i_1 j_1] \cdots [i_k j_k]$ and $h_a = 1$ for $a \neq i_s, j_s$, and

$$h_{i_s} = g_{i_s} \cdots g_{i_1} x (g_{j_s} \cdots g_{j_1})^{-1}, \qquad s = 1, \cdots, k,$$
$$h_{j_s} = g_{j_s} \cdots g_{j_1} x^{-1} (g_{i_s} \cdots g_{i_1})^{-1}, \qquad s = 1, \cdots, k.$$

Clearly $h_{i_k} = x$ and $h_{j_k} = x^{-1}$ by Eqn. (2.18). Therefore we have the following equations for $s = 1, 2, \cdots, k\,(mod\,k)$

$$h_{i_s} = g_{i_s} h_{i_{s-1}} g_{j_s}^{-1}, \quad h_{j_s} = g_{j_s} h_{j_{s-1}} g_{i_s}^{-1}, \tag{2.19}$$

which imply that $hs(\mu)(g)s(\rho)(h^{-1}) = g$. Note also that $s(\rho)s(\mu) = s(\mu)s(\sigma)$ and $d(\mu) = k\,(mod\,2)$. We see that the conjugation

(2.20)
$$(h, t_\mu)(g, t_\rho)(h, t_\mu)^{-1} = (hs(\mu)(g)s(\rho)(h^{-1}), z^k t_\rho^{s(\mu)}) = (g, z^k t_\rho^{s(\mu)}),$$

where we used $d(\rho) = 1$. Observe that by Eqn. (2.8)

$$t_\rho^{s(\mu)} = \cdots [j_1 \cdots j_k][i_1 \cdots i_k] \cdots = z^{(k-1)^2} t_\rho = z^{k-1} t_\rho.$$

Plugging this into Eqn. (2.20) we obtain that $(h, t_\mu)(g, t_\rho)(h, t_\mu)^{-1} = z(g, t_\rho)$, and this contradiction says that each partition $\rho(c)$ must be strict. $\square$

Let C_ρ be a conjugacy class in Γ_n of type $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathcal{P}_n(\Gamma_*)$. We fix an order of the conjugacy classes of Γ as before: $c^0, \cdots, c^r$. Let $T^{\rho(c^i)}$ be the special tableau such that the numbers $(\sum_{j=0}^{i-1} |\rho(c^j)|) + 1$, $\cdots$, $\sum_{j=0}^{i} |\rho(c^j)|$ appear in the natural order from the left to right and up to bottom in the Young diagram of shape $\rho(c^i)$, and thus

$$\begin{aligned} t^{\rho(c^i)} &= [1 + a_{i-1}, \cdots, \rho(c^i)_1 + a_{i-1}] \cdots \\ &\quad [\rho(c^i)_1 + \cdots + \rho(c^i)_{l-1} + a_{i-1}, \cdots, |\rho(c^i)| + a_{i-1}], \end{aligned} \tag{2.21}$$

where $a_{i-1} = \sum_{j=0}^{i-1} |\rho(c^j)|$ and $\rho(c^i) = (\rho(c^i)_1, \cdots, \rho(c^i)_l)$. We define the special element t^ρ by

$$t^\rho = t^{\rho(c^0)} t^{\rho(c^1)} \cdots t^{\rho(c^r)}. \tag{2.22}$$

For each split conjugacy class C_ρ in Γ_n of type ρ, we define the conjugacy class D_ρ^+ in $\widetilde{\Gamma}_n$ to be the conjugacy class containing the element (g, t^ρ). We also define $D_\rho^- = z D_\rho^+$. Then $\theta_n^{-1}(C_\rho) = D_\rho^+ \cup D_\rho^-$. Let $(D_\rho^+)^{-1} = \{x^{-1} | x \in D_\rho^+\}$. We remark that $(D_\rho^+)^{-1} = z^{n(\rho')} (D_{\overline{\rho}})^+$, where $n(\rho')$ is defined in (2.13) and $\overline{\rho}$ is the partition-valued function given by $\overline{\rho}(c) = \rho(c^{-1})$.

Given a partition $\lambda = (1^{m_1} 2^{m_2} \dots)$, we define

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

We note that z_λ is the order of the centralizer of an element of cycle-type λ in $S_{|\lambda|}$.

For each partition-valued function $\rho = (\rho(c))_{c \in \Gamma_*}$ we define

$$Z_\rho = \prod_{c \in \Gamma_*} z_{\rho(c)} \zeta_c^{l(\rho(c))},$$

which is the order of the centralizer of an element of conjugacy type $\rho = (\rho(c))_{c \in \Gamma_*}$ (see [M]).

Proposition 2.6. *The order of the centralizer of an element of conjugacy type ρ in $\widetilde{\Gamma}_n$ is given by*

$$\tilde{Z}_\rho = \begin{cases} 2Z_\rho, & C_\rho \text{ is split} \\ Z_\rho, & C_\rho \text{ is non-split.} \end{cases}$$

Proof. Let C_ρ be a conjugacy class in Γ_n. If $\theta^{-1}(C_\rho)$ does not split, then $\theta^{-1}(C_\rho)$ is a conjugacy class in $\widetilde{\Gamma}_n$, so its centralizer has the order $|\widetilde{\Gamma}_n|/|\theta^{-1}(C_\rho)| = |\Gamma_n|/|C_\rho| = Z_\rho$. Otherwise $\theta^{-1}(C_\rho) = D_\rho^+ \cup D_\rho^-$, and $|\widetilde{\Gamma}_n|/|D_\rho^\pm| = 2Z_\rho$. □

Theorem 2.7. [HH] *(1) The number of conjugacy classes of Γ_n and Γ_n^0 are given by the following formulas:*

$$|\text{split classes of the group } \Gamma_n| = 2|\mathcal{SP}_n^1(\Gamma_*)| + |\mathcal{SP}_n^0(\Gamma_*)|,$$
$$|\text{split classes of the group } \Gamma_n^0| = |\mathcal{SP}_n^1(\Gamma_*)| + 2|\mathcal{SP}_n^0(\Gamma_*)|.$$

(2) The number of irreducible double spin representations is equal to the number of even strict partition-valued functions on Γ_*, and the number of pairs of irreducible associate spin representations is equal to the number of odd strict partition-valued functions on Γ_*.

Proof. The first statement in Part (1) is a corollary of Theorem 2.5 and Lemma 2.3. To see the second equation in (1), we observe that an irreducible spin representation π decomposes as follows when restricting to the subgroup $\widetilde{\Gamma}_n^0$:

$$\pi|_{\widetilde{\Gamma}_n^0} = \begin{cases} \pi_1 \oplus \pi_2, & \pi \text{ is a double spin,} \\ \pi, & \pi \text{ is an associate spin.} \end{cases}$$

Moreover a pair of the associated spin representations, when restricted to $\widetilde{\Gamma}_n^0$, become the same irreducible representation. Applying the counting formulas in (1) we obtain Part (2). □

We remark that the number of split conjugacy classes of Γ_n contained in Γ_n^0 is equal to $|\mathcal{SP}_n^1(\Gamma_*)| + |\mathcal{SP}_n^0(\Gamma_*)| = |\mathcal{OP}_n(\Gamma_*)|$ by Theorem 2.5.

3. The Hopf algebra R_Γ^- of super spin characters

3.1. **Superalgebras and supermodules.** We basically follow the exposition of [Jo] in this subsection. A complex *superalgebra* $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded complex vector space with a binary product $A \times A \longrightarrow A$ such that $A_i A_j \subset A_{i+j}$.

A vector space $V = V_0 \oplus V_1$ is a *supermodule* for a superalgebra $A = A_0 \oplus A_1$ if $A_i V_j \subset V_{i+j}$. Elements of V_i are called homogeneous. A linear map $f : M \to N$ between two A-supermodules is a *super homomorphism* of degree i if $f(M_j) \subset M_{i+j}$ and for any homogeneous element $a \in A$ and any homogeneous vector $m \in M$ we have

$$f(am) = (-1)^{d(a)d(f)} a f(m).$$

Let

$$Hom_A(M, N) = Hom_A(M, N)_0 \oplus Hom_A(M, N)_1,$$

where $Hom_A(M, N)_i$ consists of A-super-homomorphisms of degree i from M to N.

Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two supermodules. The tensor product $V \otimes W$ is also a supermodule with $(V \otimes W)_i = \sum_{k+l=i(mod 2)} V_k \otimes W_l$. Submodules, irreducible or simple supermodules are defined similarly as usual. Two examples of complex simple superalgebras are given in order.

Let $r, s \in \mathbb{N}$. We define $M(r|s)$ to be the $\mathbb{C}$-superalgebra of $(r+s)$-square matrices with the grading

$$M(r|s)_0 = \{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} | A \in M_{r,r}(\mathbb{C}), D \in M_{s,s}(\mathbb{C}) \},$$

$$M(r|s)_1 = \{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} | B \in M_{r,s}(\mathbb{C}), C \in M_{s,r}(\mathbb{C}) \}$$

and the operations are the underlying usual matrix addition and multiplication. As in the ungraded case, $M(r|s)$ can also be viewed as the superalgebra of $\mathbb{Z}_2$-graded linear maps of $\mathbb{C}^{r|s} = \mathbb{C}^r \oplus \mathbb{C}^s$ with the usual superpositions of maps. It is easily seen that $M(r|s)$ is a simple $\mathbb{C}$-superalgebra and $\mathbb{C}^{r|s}$ is a simple $M(r|s)$-supermodule.

Another example is the $\mathbb{C}$-superalgebra $Q(n)$. As a supervector space it is defined by

$$Q(n)_0 = \{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} | A \in M_{n,n}(\mathbb{C}) \},$$

$$Q(n)_1 = \{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} | B \in M_{n,n}(\mathbb{C}) \}.$$

The superalgebra structure is given by the usual matrix multiplication. The space $\mathbb{C}^{n|n}$ is also a $Q(n)$-supermodule under the usual matrix multiplication.

Wall [Wl] showed that these two simple superalgebras are the only two types of simple superalgebras over $\mathbb{C}$. In the sequel we will call the supermodule $\mathbb{C}^{r|s}$ of type M if it is considered as a $M(r|s)$-supermodule and $\mathbb{C}^{n|n}$ of type Q if it is considered as a $Q(n)$-supermodule.

For any finite group G and a subgroup H of index 2, we set the parity of elements of H (resp. $G \backslash H$) to be even (resp. odd). The corresponding group superalgebra of G is semisimple (see [Jo]). In the case of the spin wreath product $\widetilde{\Gamma}_n$ and the subgroup $\widetilde{\Gamma}_n^0 = \Gamma^n \wr \tilde{A}_n$,

this parity agrees with the parity given by the homomorphism d (see (2.12). As a superalgebra, $\mathbb{C}[\widetilde{\Gamma}_n]$ is given by

$$\mathbb{C}[\widetilde{\Gamma}_n]_0 = \{\sum_g a_g g | g \in \widetilde{\Gamma}^0_n\}, \tag{3.1}$$

$$\mathbb{C}[\widetilde{\Gamma}_n]_1 = \{\sum_g a_g g | g \in \widetilde{\Gamma}^1_n\}, \tag{3.2}$$

and the product is the usual multiplication.

Proposition 3.1. *There exists an isomorphism of $\mathbb{C}$-superalgebras*

$$\mathbb{C}[\widetilde{\Gamma}_n] \simeq \bigoplus_i M(r_i|s_i) \oplus \bigoplus_j Q(n_j).$$

Any finite dimensional $\mathbb{C}[\widetilde{\Gamma}_n]$-supermodule is isomorphic to a direct sum of simple supermodules of type M and Q.

By the definition of spin representations and Lemma 2.3 we know the number of irreducible spin supermodules for $\widetilde{\Gamma}_n$.

Proposition 3.2. *The number of irreducible spin supermodules of $\widetilde{\Gamma}_n$ is equal to $|\mathcal{SP}_n(\Gamma_*)|$, the number of strict partition-valued functions on Γ_*. If V is an irreducible $\widetilde{\Gamma}_n$-supermodule of type M, then its underlying $\widetilde{\Gamma}_n$-module is irreducible. If V is an irreducible $\widetilde{\Gamma}_n$-supermodule of type Q, then its underlying $\widetilde{\Gamma}_n$-module decomposes into two irreducible $\widetilde{\Gamma}_n$-modules U and U', where $U' = U$ as a vector space and its action is given by $a.u = (-1)^{d(a)}au$ for any homogeneous element $a \in \widetilde{\Gamma}_n$.*

3.2. **Induced supermodules.** Let G be a finite group with a central involution z and a parity epimomorphism $d : G \longrightarrow \mathbb{Z}_2$ such that $d(z) = 0$. Let H be a subgroup of G containing z such that the restriction of d on H is not identically zero. Such a pair (G, H) of finite groups will be called an *admissible pair* of finite groups.

The group algebras $\mathbb{C}[G]$ and $\mathbb{C}[H]$ become superalgebras with $G^0 = ker(d), H^0 = ker(d|_H)$ and $G^1 = G\backslash G^0, H^1 = G\backslash H^0$.

Let W be a $\mathbb{C}[H]$-supermodule. We define the *induced supermodule* $Ind^G_H W$ for $\mathbb{C}[G]$ by

$$Ind^G_H W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \tag{3.3}$$

with the action given by $g(h \otimes w) = gh \otimes w$. Clearly $Ind^G_H W$ is a spin supermodule if W is a spin supermodule.

The following lemma can be checked similarly as the ordinary case [Sr].

Lemma 3.3. *Let (G, H) be an admissible pair of finite groups. Let V be a $\mathbb{C}[G]$-supermodule and W a $\mathbb{C}[H]$-sub-supermodule of $V|_H$. Then V is equal to $Ind_H^G W$ if and only if*

$$V = \bigoplus_{s \in G/H} sW, \tag{3.4}$$

where sW denotes the subspace $x.W$ $(x \in s)$ of the supermodule V.

Let (G, K) be another admissible pair of finite groups. Consider the double cosets HsK of H and K in G. For $s \in H\backslash G/K$ the set $H_s := s^{-1}Hs \cap K$ is a subgroup of K. The following analog of Mackey's theorem can be proved similarly as in the ordinary case using Lemma 3.3.

Proposition 3.4. *Let (G, H) and (G, K) be admissible pairs of finite groups as above. Then we have*

$$Res_K Ind_H^G W \simeq \bigoplus_{s \in H\backslash G/K} Ind_{H_s}^K Res_{H_s} W \tag{3.5}$$

as supermodules.

3.3. **The space $R^-(\widetilde{\Gamma}_n)$.** A *spin class function* on $\widetilde{\Gamma}_n$ is a class function map from $\widetilde{\Gamma}_n$ to $\mathbb{C}$ such that

$$f(zx) = -f(x).$$

Thus spin class functions vanish on non-split conjugacy classes. A *spin super class function* on $\widetilde{\Gamma}_n$ is a spin class function f on $\widetilde{\Gamma}_n$ such that f vanishes further on odd strict conjugacy classes. In other words, f corresponds to a complex functional on $\mathcal{OP}_n(\Gamma_*)$ in view of Theorem 2.5.

Let $R^-(\widetilde{\Gamma}_n)$ be the $\mathbb{C}$-span of spin super class functions on $\widetilde{\Gamma}_n$. Let $R(\widetilde{\Gamma}_n)$ be the $\mathbb{C}$-span of class functions on $\widetilde{\Gamma}_n$. Let $R^0(\widetilde{\Gamma}_n)$ be the subspace of the class functions $f(x)$ such that $f(zx) = f(x), x \in \widetilde{\Gamma}_n$, and let $R^1(\widetilde{\Gamma}_n)$ be the space of spin class functions. Then we have

$$R(\widetilde{\Gamma}_n) = R^0(\widetilde{\Gamma}_n) \oplus R^1(\widetilde{\Gamma}_n),$$
$$R^-(\widetilde{\Gamma}_n) \subset R^1(\widetilde{\Gamma}_n), \qquad R^0(\widetilde{\Gamma}_n) \simeq R(\Gamma_n).$$

In this paper we will focus on the space $R^-(\widetilde{\Gamma}_n)$. We remark that $R^1(\widetilde{\Gamma}_n)$ can be identified as a vector space with the Grothendieck ring of spin representations of $\widetilde{\Gamma}_n$, and it is not difficult to recover $R^1(\widetilde{\Gamma}_n)$ from $R^-(\widetilde{\Gamma}_n)$ using Proposition 3.2.

The standard inner product $\langle\ |\ \rangle$ on $R(\widetilde{\Gamma}_n)$ induces an inner product on $R^-(\widetilde{\Gamma}_n)$. For two spin super class functions $f, g \in R^-(\widetilde{\Gamma}_n)$ we define

$$\begin{aligned}\langle f, g\rangle &= \langle f, g\rangle_{\widetilde{\Gamma}_n} \\ &= \frac{1}{|\widetilde{\Gamma}_n|}\sum_{x\in\widetilde{\Gamma}_n^0} f(x)g(x^{-1}) = \frac{1}{2}\langle f, g\rangle_{\widetilde{\Gamma}_n^0},\end{aligned} \tag{3.6}$$

where $\langle f, g\rangle_{\widetilde{\Gamma}_n^0}$ is the inner product of $f|_{\widetilde{\Gamma}_n^0}$ and $g|_{\widetilde{\Gamma}_n^0}$ in the space of class functions on the subgroup $\widetilde{\Gamma}_n^0$. Since even split conjugacy classes of $\widetilde{\Gamma}_n$ have the form $\{D_\rho^+\}\cup\{D_\rho^-\}$ and $f(D_\rho^+) = -f(D_\rho^-)$, we can rewrite the inner product by using Proposition 2.6.

$$\langle f, g\rangle = \sum_{\rho\in\mathcal{OP}_n(\Gamma_*)} \frac{1}{Z_\rho} f(\rho)g(\overline{\rho}), \tag{3.7}$$

where $f(\rho) = f(D_\rho^+)$ and $\overline{\rho}$ is defined in Sect. 2.4. In the sequel we will fix the value of a class function at $\rho \in \mathcal{OP}_n(\Gamma_*)$ to be the value at the conjugacy class D_ρ^+.

Let $V = V_0 \oplus V_1$ be a $\widetilde{\Gamma}_n$-supermodule. We define the character of V as the function $\chi_V : x \mapsto tr(x)$, $x \in \widetilde{\Gamma}_n$. Clearly $\chi_V(\widetilde{\Gamma}_n^1) = 0$.

Proposition 3.5. *The characters of irreducible spin $\widetilde{\Gamma}_n$-supermodules form a $\mathbb{C}$-basis of $R^-(\widetilde{\Gamma}_n)$. Let ϕ and γ be two irreducible characters of spin supermodules, then*

$$\langle \phi, \gamma\rangle = \begin{cases} 1 & \text{if } \phi \simeq \gamma, \text{ type } M \\ 2 & \text{if } \phi \simeq \gamma, \text{ type } Q \\ 0 & \text{otherwise}\end{cases}. \tag{3.8}$$

Conversely, if $\langle f, f\rangle = 1$ for $f \in R^-(\widetilde{\Gamma}_n)$, then $\pm f$ affords an irreducible spin $\widetilde{\Gamma}_n$-supermodule of type M. If $\langle f, f\rangle = 2$, then either $\pm f$ is the character of an irreducible spin supermodule of type Q or f is a sum or difference of two irreducible characters of spin supermodules of type M.

Proof. Let ξ_c be the characteristic function on the conjugacy class c. Then $R^-(\widetilde{\Gamma}_n)$ is spanned by ξ_c, where c ranges over the set of split even classes. Thus $dim(R^-(\widetilde{\Gamma}_n)) \leq |\mathcal{OP}_n(\Gamma_*)|$.

On the other hand we see that the characters of spin supermodules are class functions in $R^-(\widetilde{\Gamma}_n)$ since the trace of any odd endomorphism is zero. Let ϕ and γ be the characters of two irreducible spin supermodules of $\widetilde{\Gamma}_n$. It follows from Proposition 3.2 that the underlying

module of ϕ or γ is either irreducible module or the sum of two associated irreducible modules according to their types, which implies immediately the orthogonality relation (3.8). Therefore the matrix of the inner product is orthogonal on the set of super spin characters. Then by Lemma 2.3 and Proposition 3.2

$$dim(R^-(\widetilde{\Gamma}_n)) \geq |\mathcal{SP}_n(\Gamma_*)| = |\mathcal{OP}_n(\Gamma_*)|.$$

Thus the two inequalities above become equality, and so the irreducible characters of spin $\widetilde{\Gamma}_n$-supermodules form a $\mathbb{Z}$-basis in $R^-(\widetilde{\Gamma}_n)$.

The last characterization of irreducible supermodules follows from the semi-simplicity of the superalgebra $\mathbb{C}[\widetilde{\Gamma}_n]$ and the usual orthogonality of ordinary irreducible characters. $\square$

3.4. **Hopf algebra structure on R^-_Γ.** We now define one of our main objects

$$R^-_\Gamma = \bigoplus_{n\geq 0} R^-(\widetilde{\Gamma}_n).$$

Let $\widetilde{\Gamma}_n \tilde{\times} \widetilde{\Gamma}_m$ be the direct product of $\widetilde{\Gamma}_n$ and $\widetilde{\Gamma}_m$ with a twisted multiplication

$$(t, t') \cdot (s, s') = (tsz^{d(t')d(s)}, t's'),$$

where $s, t \in \widetilde{\Gamma}_n, s', t' \in \widetilde{\Gamma}_m$ are homogeneous. Note that $|\widetilde{\Gamma}_n \tilde{\times} \widetilde{\Gamma}_m| = |\widetilde{\Gamma}_n||\widetilde{\Gamma}_m|$. We define the *spin product* of $\widetilde{\Gamma}_n$ and $\widetilde{\Gamma}_m$ (see [HH]) by

$$\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m = \widetilde{\Gamma}_n \tilde{\times} \widetilde{\Gamma}_m / \{(1,1), (z,z)\}, \tag{3.9}$$

which can be embedded into the spin group $\widetilde{\Gamma}_{n+m}$ canonically by letting

$$((g, t'_i), 1) \mapsto (g, t_i), \qquad (1, (g, t''_j)) \mapsto (g, t_{n+j}), \tag{3.10}$$

where $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$. We will identify $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$ with its image in $\widetilde{\Gamma}_{m+n}$ and regard it as a subgroup of $\widetilde{\Gamma}_{m+n}$. Clearly $\theta_{n+m}(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m)$ is the pull-back of $\Gamma_n \times \Gamma_m$.

Remark 3.6. Partition $\{1, 2, \cdots, n+m\}$ into a disjoint union of subsets I and J with $|I| = n$ and $|J| = m$. Then $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$ can be embedded into $\widetilde{\Gamma}_{n+m}$ using a similar map as (3.10) by mapping the generators of $\widetilde{S}_n$ and $\widetilde{S}_m$ to the generators of $\widetilde{S}_{n+m}$ indexed by I and J respectively. One can check that all such embeddings of $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$ are conjugate subgroups in $\widetilde{\Gamma}_{n+m}$.

The subgroup $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$ has a distinguished subgroup of index 2 consisting of even elements given by d. We define $R^-(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m)$ to be the

space of spin class functions on $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$ that vanish on odd conjugacy classes of $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$.

For two spin supermodules U and V of $\widetilde{\Gamma}_n$ and $\widetilde{\Gamma}_m$ we define the *super (outer)-tensor product* $U \otimes V$ by

$$(t, s) \cdot (u \otimes v) = (-1)^{d(s)d(u)}(tu \otimes sv),$$

where s and u are homogeneous elements. We see immediately that

$$\begin{aligned}(z', z'') \cdot (u \otimes v) &= (-u) \otimes (-v) = u \otimes v, \\ (z, 1) \cdot (u \otimes v) &= -(u \otimes v).\end{aligned}$$

This says that $U \otimes V$ is a spin $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$-supermodule.

The following is a direct generalization of a result in [Jo] for trivial Γ.

Proposition 3.7. *Let U and V be simple supermodules for $\widetilde{\Gamma}_n$ and $\widetilde{\Gamma}_m$ respectively. Then*
1) If both U and V are of type M, then $U \otimes V$ is a simple $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$-supermodule of type M.
2) If U and V are of different type, then $U \otimes V$ is a simple $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$-supermodule of type Q.
3) If both U and V are type Q, then $U \otimes V \simeq N \oplus N$ for some simple $\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m$-supermodule N of type M.

The super (outer)tensor product defines an isometric isomorphism

$$R^-(\widetilde{\Gamma}_n) \bigotimes R^-(\widetilde{\Gamma}_m) \stackrel{\phi_{n,m}}{\to} R^-(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m), \tag{3.11}$$

which is actually an isomorphism over $\mathbb{Q}$ by Proposition 3.7.

The space R^-_Γ carries a multiplication defined by the composition

$$m : R^-(\widetilde{\Gamma}_n) \bigotimes R^-(\widetilde{\Gamma}_m) \xrightarrow{\phi_{n,m}} R^-(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m) \xrightarrow{Ind} R^-(\widetilde{\Gamma}_{n+m}), \tag{3.12}$$

and a comultiplication defined by the composition

$$\begin{aligned}\Delta : R^-(\widetilde{\Gamma}_n) &\xrightarrow{Res} \bigoplus_{m=0}^{n} R^-(\widetilde{\Gamma}_{n-m} \hat{\times} \widetilde{\Gamma}_m) \\ &\xrightarrow{\phi^{-1}} \bigoplus_{m=0}^{n} R^-(\widetilde{\Gamma}_{n-m}) \bigotimes R^-(\widetilde{\Gamma}_m).\end{aligned} \tag{3.13}$$

Here Ind and Res denote the induction (see (3.3) and restriction functors respectively. The isomorphism ϕ^{-1} is equal to $\oplus_{0 \le m \le n} \phi^{-1}_{n-m,m}$ (see (3.11)).

Theorem 3.8. *The above operations define a Hopf algebra structure for R^-_Γ.*

Proof. Using Remark 3.6 twice we observe that the following two embeddings give rise to two conjugate subgroups in $\widetilde{\Gamma}_{n+m+l}$ (see Remark 3.6):

$$(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m) \hat{\times} \widetilde{\Gamma}_l \hookrightarrow \widetilde{\Gamma}_{n+m+l} \hookleftarrow \widetilde{\Gamma}_n \hat{\times} (\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_l).$$

Using this and Lemma 3.3 we can easily check the associativity of the product.

For a simple supermodule V we define

$$c(V) = \begin{cases} 0 & V \text{ is type M} \\ 1 & V \text{ is type Q} \end{cases}. \tag{3.14}$$

Let V_1, V_2, and V_3 be simple supermodules for $\mathbb{C}[\widetilde{\Gamma}_n]$, $\mathbb{C}[\widetilde{\Gamma}_m]$, and $\mathbb{C}[\widetilde{\Gamma}_l]$ respectively. It is easy to see that $c(V_1, V_2) = c(V_1)c(V_2)$ satisfies the cocycle condition

$$c(V_1, V_2) + c(V_1 \otimes V_2, V_3) = c(V_2, V_3) + c(V_1, V_2 \otimes V_3). \tag{3.15}$$

Therefore we can define $c(V_1 \otimes V_2 \otimes V_3)$ to be either of the above expressions.

Using the cocycle c we prove the coassociativity as follows. Let U be a $\mathbb{C}[\widetilde{\Gamma}_n]$-supermodule and suppose that $Res_{\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_l \hat{\times} \widetilde{\Gamma}_k} U = \oplus_i U_i(m, l, k)$ as an irreducible decomposition. Then we have

$$\begin{aligned}
(1 \otimes \Delta)\Delta(U) &= (1 \otimes \phi^{-1})\phi^{-1} \bigoplus_{m+l+k=n} Res_{\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_l \hat{\times} \widetilde{\Gamma}_k} U \\
&= \bigoplus_{m+l+k=n,i} 2^{-c(U_i(m,l,k))} U_i(m, l, k) \\
&= (\phi^{-1} \otimes 1)\phi^{-1} \bigoplus_{m+l+k=n} Res_{\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_l \hat{\times} \widetilde{\Gamma}_k} U \\
&= (\Delta \otimes 1)\Delta(U),
\end{aligned}$$

where we used the cocycle condition in the third equation, and the notation $\sum 2^{c(U_i)} U_i$ stands for the multiplicity-free summation of the irreducible components (c.f. Proposition 3.7 and definition of ϕ).

Finally we look at the compatibility of multiplication and comultiplication. Fix m and n, it follows from Proposition 3.4 that

$$\begin{aligned}
\Delta(U \cdot V) &= \bigoplus_{k+l=m+n} \phi_{k,l}^{-1} Res_{\widetilde{\Gamma}_k \hat{\times} \widetilde{\Gamma}_l} Ind_{\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_n}^{\widetilde{\Gamma}_{m+n}} \phi_{m,n}(U \otimes V) \\
&= \bigoplus_{k+l=m+n} \bigoplus_s \phi_{k,l}^{-1} Ind_{(\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_n)_s}^{\widetilde{\Gamma}_k \hat{\times} \widetilde{\Gamma}_l} Res_{(\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_n)_s} \phi_{m,n}(U \otimes V)_s,
\end{aligned}$$

where s runs through the double cosets $\widetilde{\Gamma}_m \hat{\times} \widetilde{\Gamma}_n \backslash \widetilde{\Gamma}_{m+n} / \widetilde{\Gamma}_k \hat{\times} \widetilde{\Gamma}_l$. Notice that the double cosets are in one-to-one correspondence with the double cosets $\Gamma_m \times \Gamma_n \backslash \Gamma_{m+n} / \Gamma_k \times \Gamma_l$. Again by the cocycle property of c and counting the double cosets we can check that the last summation is exactly $\Delta(U) \cdot \Delta(V)$. □

Remark 3.9. Our Hopf algebra is different from that of [HH] where a bigger space than our R_Γ^- was used.

The standard bilinear form in R_Γ^- is defined in terms of those on $R^-(\widetilde{\Gamma}_n)$ as follows:

$$\langle u, v \rangle = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\widetilde{\Gamma}_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in \widetilde{\Gamma}_n$.

4. Basic spin representations of $\widetilde{\Gamma}_n$

4.1. **A weighted bilinear form on $R(\Gamma)$ and $R^-(\widetilde{\Gamma}_n)$.** In [FJW1] we introduced the notion of weighted bilinear forms on $R(\Gamma)$ and coherently combined several examples in this concept. We will also similarly define weighted bilinear forms on the space $R^-(\widetilde{\Gamma}_n)$.

Let ξ be a self-dual class function in $R(\Gamma)$, i.e. $\xi(c) = \xi(c^{-1})$. Let $*$ denote the product of two characters in $R(\Gamma)$, which is afforded by the tensor product. Let $a_{ij} \in \mathbb{C}$ be the (virtual) multiplicities of γ_j in $\xi * \gamma_i$:

$$\xi * \gamma_i = \sum_{j=0}^{r} a_{ij} \gamma_j. \tag{4.1}$$

We denote further by A the $(r+1) \times (r+1)$ matrix $(a_{ij})_{0 \leq i,j \leq r}$. Then the weighted bilinear form $\langle f, g \rangle_\xi$ is defined by

$$\langle f, g \rangle_\xi = \langle \xi * f, g \rangle_\Gamma, \quad f, g \in R(\Gamma).$$

Alternatively it can be explicitly given by

$$\begin{aligned} \langle f, g \rangle_\xi &= \frac{1}{|\Gamma|} \sum_{x \in \Gamma} \xi(x) f(x) g(x^{-1}) \\ &= \sum_{c \in \Gamma_*} \zeta_c^{-1} \xi(c) f(c) g(c^{-1}) && (4.2) \\ &= \sum_{c \in \Gamma_*} \zeta_c^{-1} \xi(c) f(c^{-1}) g(c). && (4.3) \end{aligned}$$

In particular, Eqn. (4.1) is equivalent to

$$\langle \gamma_i, \gamma_j \rangle_\xi = a_{ij}. \tag{4.4}$$

The self-duality implies that A is a symmetric matrix. Note that the weighted bilinear form becomes the standard bilinear form when $\xi = \gamma_0$, the trivial character of Γ.

Let V be a spin supermodule for $\widetilde{\Gamma}_n$ and W a module for Γ_n. As a $\mathbb{Z}_2$-graded vector space $W \otimes V = W \otimes V_0 \oplus W \otimes V_1$ and the action of $\widetilde{\Gamma}_n$ is defined by

$$(g, z^p t_\rho)(w \otimes v) = (g, s(\rho)) \cdot w \otimes (g, z^p t_\rho) \cdot v, \qquad g \in \Gamma^n, \sigma \in \mathcal{P}_n(\Gamma_*). \tag{4.5}$$

It is easy to check that the tensor product $V \otimes W$ is a spin $\widetilde{\Gamma}_n$-supermodule. This construction defines a morphism:

$$R(\Gamma_n) \otimes R^-(\widetilde{\Gamma}_n) \xrightarrow{\ *\ } R^-(\widetilde{\Gamma}_n). \tag{4.6}$$

Let us recall the construction of character $\eta_n(\xi)$ in [W, FJW1]. Let γ be an irreducible character of Γ afforded by the Γ-module V, the tensor product $V^{\otimes n}$ is naturally a Γ_n-module by the direct product action of Γ^n composed with permutation action of the symmetric group S_n. The resulting character of Γ_n is denoted by $\eta_n(\gamma)$. Furthermore we can extend η_n from Γ^* to $R(\Gamma)$. The character value of $\eta_n(\xi)$ at the class $\rho = (\rho(c))$ is given by

$$\eta_n(\xi)(\rho) = \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}. \tag{4.7}$$

It is clear that the class function $\eta_n(\xi)$ is self-dual as long as ξ is.

We now introduce a *weighted bilinear form* on $R^-(\widetilde{\Gamma}_n)$ by letting

$$\langle f, g \rangle_{\xi, \widetilde{\Gamma}_n} = \langle \eta_n(\xi) * f, g \rangle_{\widetilde{\Gamma}_n}, \quad f, g \in R^-(\widetilde{\Gamma}_n),$$

where we used the map (4.6). The self-duality of $\eta_n(\xi)$ implies that the bilinear form $\langle\ ,\ \rangle_\xi$ is symmetric.

Remark 4.1. When $n = 1$, this weighted bilinear form obviously reduces to the weighted bilinear form on $R(\Gamma)$ defined in (4.2-4.3).

The bilinear form on $R_\Gamma^- = \bigoplus_n R^-(\widetilde{\Gamma}_n)$ is given by

$$\langle u, v \rangle_\xi = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\xi, \widetilde{\Gamma}_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in R^-(\widetilde{\Gamma}_n)$.

4.2. **Basic spin representations.** Let the Pauli spin matrices be

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\sigma_2 = \begin{bmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let C_{2k} be the Clifford algebra generated by $e_1, e_2, \cdots e_{2k}$ with relations:

$$\{e_i, e_j\} = e_i e_j + e_j e_i = -2\delta_{ij}. \tag{4.8}$$

Thus $e_j^2 = -1$. The Clifford algebra C_{2k} is endowed with a natural superalgebra structure by letting the parity of e_i to be odd for each i. When $k = 1$, one has that $C_2 \simeq M(1|1) = End(\mathbb{C}^{1|1})$ and the action of C_2 on $\mathbb{C}^{1|1}$ is given by the Pauli spin matrices:

$$e_1 \mapsto \sqrt{-1}\sigma_1, \qquad e_2 \mapsto \sqrt{-1}\sigma_2.$$

More generally we have $C_{2k} = End(\otimes^k \mathbb{C}^{1|1}) \simeq M(2^{k-1}|2^{k-1})$. The tensor product $\otimes^k \mathbb{C}^{1|1}$ admits a canonical supermodule structure for the Clifford algebra C_{2k} under the action

$$e_{2j-1} \longrightarrow \sqrt{-1}\sigma_3^{\otimes(j-1)} \otimes \sigma_1 \otimes \sigma_0^{\otimes(k-j)}, \quad j = 1, \dots, k, \tag{4.9}$$

$$e_{2j} \longrightarrow \sqrt{-1}\sigma_3^{\otimes(j-1)} \otimes \sigma_2 \otimes \sigma_0^{\otimes(k-j)}, \quad j = 1, \dots, k. \tag{4.10}$$

The above formulas define explicitly the structure of a simple C_{2k}-supermodule on $\otimes^k \mathbb{C}^{1|1}$.

Let C_{2k+1} be the Clifford algebra generated by e_i, $i = 1, \dots, 2k+1$ with similar relations like (4.8). We embed C_1 into $End(\mathbb{C}^{1|1})$ by $1 \mapsto Id$, $e_1 \mapsto \sqrt{-1}\sigma_1$. Then

$$C_{2k+1} \simeq C_{2k} \otimes C_1 \hookrightarrow End(\otimes^{k+1} \mathbb{C}^{1|1})$$

gives a C_{2k+1}-supermodule structure on $\otimes^{k+1}\mathbb{C}^{1|1}$. The explicit action is given by the same formulae (4.9-4.10), except that $j = 1, \cdots, k+1$ in (4.9). Observe that $C_{2k+1} \simeq Q(2^k)$.

It is well-known (see e.g. [Jo]) that there exists an embedding of $\widetilde{S}_n$ into the multiplicative Clifford group of units in C_{n-1}. Therefore $\otimes^{[\frac{n}{2}]}\mathbb{C}^{1|1}$ can be regarded as a $\widetilde{S}_n$-supermodule, which is called the *basic spin supermodule* for $\widetilde{S}_n$. More explicitly we have

Proposition 4.2. [S, Jo] *The basic spin supermodule for $\widetilde{S}_n$ is $\otimes^{[\frac{n}{2}]}\mathbb{C}^{1|1}$ with the action*

$$t_j \mapsto \sqrt{\frac{j+1}{2j}} e_j - \sqrt{\frac{j-1}{2j}} e_{j-1}, \qquad j = 1, \cdots, n-1. \tag{4.11}$$

Here we take $e_0 = 0$. Its character χ_n is given by

$$\chi_n(\alpha) = \begin{cases} 2^{l(\alpha)/2} & \text{if } \alpha \in \mathcal{OP}_n,\ n \text{ even} \\ 2^{(l(\alpha)-1)/2} & \text{if } \alpha \in \mathcal{OP}_n,\ n \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

In particular $\chi_n(1) = 2^{\lceil \frac{n}{2} \rceil}$. Here $\lceil a \rceil$ denotes the largest integer $\leq a$.

Proposition 4.3. [S, Mo] *1) Let $n \geq 1$ be an odd integer. The basic spin supermodule $\otimes^{(n-1)/2}\mathbb{C}^{1|1}$ is an irreducible $\widetilde{S}_n$-module under the action (4.11). Its character χ_n is given by the second equation of (4.12). In particular $\chi_n(1) = 2^{(n-1)/2}$.*

2) Let $n \geq 1$ be an even integer. The basic spin supermodule is a reducible $\widetilde{S}_n$-module under the action (4.11) and decomposes into two irreducible $\widetilde{S}_n$-modules whose characters $\chi_n^{\pm}$ are given by

$$\chi_n^{\pm}(\alpha) = \begin{cases} 2^{(l(\alpha)-2)/2} & \text{if } \alpha \in \mathcal{OP}_n, \\ \pm(\sqrt{-1})^{n/2}\sqrt{\frac{n}{2}} & \text{if } \alpha = (n), \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

In particular, $\chi_n^{\pm}(1) = 2^{(n-2)/2}$.

4.3. **The spin character $\pi_n(\gamma)$ of $\widetilde{\Gamma}_n$.** Let V be a Γ-module afforded by the character $\gamma \in R(\Gamma)$, and let U be a spin supermodule (resp. module) of $\widetilde{S}_n$ with the character π. The tensor product $V^{\otimes n} \otimes U$ has a canonical spin supermodule (resp. module) structure for $\widetilde{\Gamma}_n$ as follows (compare (4.5)). For any $g = (g_1, \cdots, g_n) \in \Gamma^n$ let $(g, z^p t_\rho)$ be an element in $\widetilde{\Gamma}_n$. The supermodule (resp. module) structure is defined by

$$\begin{aligned} &(g, z^p t_\rho).(v_i \otimes \cdots \otimes v_n \otimes u) \\ &\qquad = g_1 v_{s(\rho)^{-1}(1)} \otimes \cdots g_n v_{s(\rho)^{-1}(n)} \otimes (z^p t_\rho u). \end{aligned}$$

We denote by $\pi_n(\gamma)$ the character of the constructed spin supermodule (resp. module).

Recall that the conjugacy class D_ρ^+ contains an element (g, t^ρ) (see (2.3)).

Proposition 4.4. *Let π be the character of a spin $\widetilde{S}_n$-supermodule. Then the character values of $\pi_n(\gamma)$ at the conjugacy classes $D_\rho^{\pm}$ ($\rho \in \mathcal{OP}_n(\Gamma_*)$) are given by*

$$\pi_n(\gamma)(D_\rho^{\pm}) = \pm\pi(t^\rho) \prod_{c \in \Gamma_*} \gamma(c)^{l(\rho(c))}. \tag{4.14}$$

Proof. Consider $(g, z^p t_\rho) \in \widetilde{\Gamma}_n$, where $g = (g_1, \ldots, g_n) \in \Gamma^n$ and t_ρ is an n-cycle, say $t_\rho = [12\ldots n]$. Denote by $e_1, \ldots, e_k$ a basis of V_γ, and we write $ge_j = \sum_i a_{ij}(g)e_i$, $a_{ij}(g) \in \mathbb{C}$. It follows that

$$\begin{aligned}&(g, z^p t_\rho)(e_{j_1} \otimes \ldots \otimes e_{j_n} \otimes u)\\ &\qquad = g_1(e_{j_n}) \otimes g_2(e_{j_1}) \ldots \otimes g_n(e_{j_{n-1}}) \otimes z^p t_\rho(u),\end{aligned}$$

Thus we obtain

$$\begin{aligned}\pi_n(\gamma)(z^p t_\rho) &= \text{trace } a(g_n)a(g_{n-1})\ldots a(g_1)\pi(z^p t_\rho)\\ &= \text{trace } a(g_n g_{n-1}\ldots g_1)\pi(z^p t_\rho) = \gamma(c)\pi(z^p t_\rho),\end{aligned}$$

where we notice that $g_n g_{n-1}\ldots g_1$ lies in $c \in \Gamma_*$.

Given $x \hat{\times} y \in \widetilde{\Gamma}_n$, where $x \in \widetilde{\Gamma}_r$ and $y \in \widetilde{\Gamma}_{n-r}$, we clearly have $\pi_n(\gamma)(x \hat{\times} y) = \pi_n(\gamma)(x)\pi_n(\gamma)(y)$. Thus it follows that for the conjugacy class $D_\rho^+ \in \widetilde{\Gamma}_n$ of type ρ, we have

$$\pi_n(\gamma)(D_\rho^\pm) = \pm\pi(t^\rho)\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho(c)},$$

where $||\rho|| = n$. ∎

Since the sign character is trivial at even classes, we can extend naturally π_n to a map from $R(\Gamma)$ to $R^-(\widetilde{\Gamma}_n)$ (compare with [W, FJW1]). For two Γ-characters β and γ we define.

$$(4.15)\quad \pi_n(\beta - \gamma) = \sum_{m=0}^{n}(-1)^m Ind_{\widetilde{\Gamma}_{n-m}\hat{\times}\widetilde{\Gamma}_m}^{\widetilde{\Gamma}_n}[\pi_{n-m}(\beta) \otimes \pi_m(\gamma)].$$

When n is even, the character χ_n of the basic spin supermodule (see Sect. 4.2) decomposes into the sum of irreducible characters $\chi_n^\pm$ of $\widetilde{\Gamma}_n$-modules. For each $c \in \Gamma_*$, we define the special partition-valued function $c^{(n)} \in \mathcal{P}(\Gamma_*)$ such that

$$(4.16)\qquad c^{(n)}(c) = (n), \qquad c^{(n)}(c') = \emptyset, \quad \text{for } c' \neq c.$$

The following corollary is an immediate consequence of Propositions 4.4 and 4.3.

Corollary 4.5. *(1) The character value of $\chi_n(\gamma)$ at the conjugacy class D_ρ^+ of type ρ is*

$$(4.17)\qquad \chi_n(\gamma)(\rho) = \begin{cases} 2^{(l(\rho)-\overline{n})/2}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho(c))} & \rho \in \mathcal{OP}_n(\Gamma_*)\\ 0 & \textit{otherwise}\end{cases},$$

where $\overline{n}$ is 0 or 1 depending on whether n is even or odd.

(2) Let n be an even positive integer. The character values of $\chi_n^{\pm}(\gamma)$ at the conjugacy class D_ρ^+ of type ρ are

$$\chi_n^{\pm}(\gamma)(\rho) = \begin{cases} 2^{(l(\rho)-2)/2}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho(c))} & \rho \in \mathcal{OP}_n(\Gamma_*) \\ \pm(\sqrt{-1})^{n/2}\sqrt{\frac{n}{2}}\gamma(c) & \rho = c^{(n)} \\ 0 & \textit{otherwise} \end{cases}. \tag{4.18}$$

4.4. **Two specializations.** Let $d_i = \gamma_i(c^0)$ be the dimension of the irreducible representation of Γ afforded by the character γ_i. Let A be the matrix of the bilinear form $\langle\ |\ \rangle$ on $R(\Gamma)$ with respect to the basis γ_i. Observe that the vector

$$v_i = (\gamma_0(c^i), \gamma_1(c^i), \dots, \gamma_r(c^i))^t, \qquad (i = 0, \dots, r)$$

is an eigenvector of the matrix A with eigenvalue $\xi(c^i)$.

Two special choices of the weight function ξ will be our prototypical examples. The first choice is that $\xi = \gamma_0$, the trivial character.

Let π be the character of the 2-dimensional representation of Γ given by the embedding of Γ in $SL_2(\mathbb{C})$. Let

$$\xi = 2\gamma_0 - \pi.$$

Then the weighted bilinear form $\langle\ ,\ \rangle_\xi$ on R_Γ becomes positive semi-definite. The radical of this bilinear form is one-dimensional and spanned by the character of the regular representation of Γ

$$\delta = \sum_{i=0}^{r} d_i\gamma_i.$$

The following is the well-known list of finite subgroups of $SL_2(\mathbb{C})$: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. McKay observed that they are in one-to-one correspondence to simply-laced Dynkin diagrams of affine types [Mc]: $a_{ii} = 2$ for all i; if $\Gamma \neq \mathbb{Z}/2\mathbb{Z}$ and $i \neq j$ then $a_{ij} = 0$ or -1. If $\Gamma = \mathbb{Z}/2\mathbb{Z}$ then $a_{01} = -2$.

5. Twisted Heisenberg algebras and $\widetilde{\Gamma}_n$

5.1. **Twisted Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$.** Let $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ be the infinite dimensional Heisenberg algebra over $\mathbb{C}$, associated with a finite group Γ and a self-dual class function $\xi \in R(\Gamma)$, with generators $a_m(\gamma), m \in 2\mathbb{Z}+1, \gamma \in \Gamma^*$ and a central element C subject to the relations:

$$[a_m(\gamma), a_n(\gamma')] = \frac{m}{2}\delta_{m,-n}\langle\gamma,\gamma'\rangle_\xi C, \quad m,n \in 2\mathbb{Z}+1,\ \gamma,\gamma' \in \Gamma^*. \tag{5.1}$$

We extend $a_m(\gamma)$ to all $\gamma = \sum_{i=0}^r s_i\gamma_i \in R(\Gamma)$ ($s_i \in \mathbb{C}$) by linearity: $a_m(\gamma) = \sum_i s_i\, a_m(\gamma_i)$.

The Heisenberg algebra may contain a large center because the bilinear form $\langle \cdot, \cdot \rangle_\xi$ may be degenerate. The center of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ is spanned by C together with $a_m(\gamma), m \in 2\mathbb{Z}+1, \gamma \in R^0$, the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R(\Gamma)$.

For $m \in 2\mathbb{Z}+1, c \in \Gamma_*$ we introduce another basis for $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$:

$$a_m(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1}) a_m(\gamma). \tag{5.2}$$

The orthogonality of the irreducible characters of Γ (2.11) implies that

$$a_m(\gamma) = \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_m(c).$$

Proposition 5.1. *The commutation relations among the new basis for $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ are given by*

$$[a_m(c'^{-1}), a_n(c)] \;=\; \frac{m}{2}\delta_{m,-n}\delta_{c',c}\zeta_c\xi(c)C, \quad c, c' \in \Gamma_*,$$

where $m, n \in 2\mathbb{Z}+1$.

Proof. The proof is similar to the untwisted case [FJW1]. $\square$

5.2. **Action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ on S_Γ^- and $\overline{S}_\Gamma$.** Denote by S_Γ^- the symmetric algebra generated by $a_{-n}(\gamma), n \in 2\mathbb{Z}_+ + 1, \gamma \in \Gamma^*$. There is a natural degree operator on S_Γ^-

$$\deg(a_{-n}(\gamma)) = n, \qquad n \in 2\mathbb{Z}_+ + 1,$$

which makes S_Γ^- into a $\mathbb{Z}_+$-graded space.

We define an action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ on S_Γ^- as follows: $a_{-n}(\gamma), n > 0$ acts as a multiplication operator on S_Γ^- and C as the identity operator; $a_n(\gamma)$, $n > 0$ acts as a derivation of the symmetric algebra

$$\begin{aligned} &a_n(\gamma).a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\dots a_{-n_k}(\alpha_k) \\ =& \sum_{i=1}^k \delta_{n,n_i}\langle \gamma, \alpha_i\rangle_\xi a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\dots \check{a}_{-n_i}(\alpha_i)\dots a_{-n_k}(\alpha_k). \end{aligned}$$

Here $n_i > 0, \alpha_i \in R(\Gamma)$ for $i = 1, \dots, k$, and $\check{a}_{-n_i}(\alpha_i)$ means the very term is deleted. In other word, the operator $a_n(\gamma), n > 0, \gamma \in R^0$ acts as 0, and $a_n(\gamma), n > 0, \gamma \in R(\Gamma) - R^0$ acts as certain non-zero differential operator. Note that S_Γ^- is not an irreducible representation over $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ in general since the bilinear form $\langle\ ,\ \rangle_\xi$ may be degenerate.

Denote by S_Γ^0 the ideal in the symmetric algebra S_Γ^- generated by $a_{-n}(\gamma), n \in \mathbb{N}, \gamma \in R^0$. Denote by $\overline{S}_\Gamma$ the quotient S_Γ^-/S_Γ^0. It follows from the definition that S_Γ^0 is a subrepresentation of S_Γ^- over the

Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$. In particular, this induces a Heisenberg algebra action on $\overline{S}_\Gamma$ which is irreducible. The unit 1 in the symmetric algebra S_Γ^- is the highest weight vector. We will also denote by 1 its image in the quotient $\overline{S}_\Gamma$.

5.3. **The bilinear form on S_Γ^-.** The space S_Γ^- admits a bilinear form $\langle \ , \ \rangle'_\xi$ determined by

$$\langle 1, 1\rangle'_\xi = 1, \quad a_n(\gamma)^* = a_{-n}(\gamma), \qquad n \in 2\mathbb{Z}+1. \tag{5.3}$$

Here $a_n(\gamma)^*$ denotes the adjoint of $a_n(\gamma)$.

For any partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{OP}$ and $\gamma \in \Gamma^*$, we define

$$a_{-\lambda}(\gamma) = a_{-\lambda_1}(\gamma)a_{-\lambda_2}(\gamma)\dots.$$

For $\rho = (\rho(\gamma))_{\gamma\in\Gamma^*} \in \mathcal{OP}(\Gamma^*)$, we define

$$a_{-\rho} = \prod_{\gamma\in\Gamma^*} a_{-\rho(\gamma)}(\gamma).$$

It is clear that $a_{-\rho}, \rho \in \mathcal{OP}(\Gamma^*)$ form a basis for S_Γ^-.

Similarly we define

$$a_{-\lambda}(c) = a_{-\lambda_1}(c)a_{-\lambda_2}(c)\dots$$

for any partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{OP}$ and $c \in \Gamma_*$. For any $\rho = (\rho(c))_{c\in\Gamma_*} \in \mathcal{OP}(\Gamma_*)$, we further define

$$a'_{-\rho} = \prod_{c\in\Gamma_*} a_{-\rho(c)}(c).$$

The elements $a'_{-\rho}, \rho \in \mathcal{OP}(\Gamma_*)$ provide a new $\mathbb{C}$-basis for S_Γ^-.

Recall that $\overline{\rho} \in \mathcal{OP}(\Gamma_*)$ is given by assigning to $c \in \Gamma_*$ the partition $\rho(c^{-1})$, which is the composition of ρ with the involution on Γ_* given by $c \mapsto c^{-1}$. It follows from Proposition 5.1 that

$$\langle a'_{-\rho'}, a'_{-\overline{\rho}}\rangle'_\xi = \delta_{\rho',\rho}\frac{Z_\rho}{2^{l(\rho)}}\prod_{c\in\Gamma_*}\xi(c)^{l(\rho(c))}, \quad \rho', \rho \in \mathcal{OP}(\Gamma_*). \tag{5.4}$$

Remark 5.2. S_Γ^0 can be characterized as the radical of the bilinear form $\langle \ , \ \rangle'_\xi$ in S_Γ^-. Thus the bilinear form $\langle \ , \ \rangle'_\xi$ induces a bilinear form on $\overline{S}_\Gamma$ which will be denoted by the same notation.

6. Isometry between R_Γ^- and S_Γ^-

6.1. **The characteristic map ch.** We define a $\mathbb{C}$-linear map ch : $R_\Gamma^- \longrightarrow S_\Gamma^-$ by letting

$$ch(f) = \sum_{\rho \in \mathcal{OP}(\Gamma_*)} \frac{2^{l(\rho)/2}}{Z_\rho} f(\rho) a'_{-\overline{\rho}}, \tag{6.1}$$

where $f(\rho) = f(D_\rho^+)$. The map ch is called the *characteristic map* (compare with [S, Jo] for Γ trivial).

Fix $n \in 2\mathbb{Z}_+ + 1$ in this paragraph. Denote by $D_n(c)^+, (c \in \Gamma_*)$ the conjugacy class in $\widetilde{\Gamma}_n$ of elements $(x, t_s) \in \Gamma_n$ such that s is an n-cycle and the cycle product of (x, t_s) is c. Then set $D_n(c)^- = zD_n(c)^+$. Thus $D_n(c)^\pm$ are the associated split conjugacy classes of type $c^{(n)}$ (see (4.16)). Denote by $\sigma_n(c)$ the super class function on $\widetilde{\Gamma}_n$ which takes value $\pm\frac{n}{\sqrt{2}}\zeta_c$ on elements in the conjugacy classes $D_n(c)^\pm$, and 0 elsewhere. For $\rho = \{i^{m_i(c)}\} \in \mathcal{OP}_n(\Gamma_*)$, $\sigma_\rho = \prod_{i \in 2\mathbb{Z}_+ + 1, c \in \Gamma_*} \sigma_i(c)^{m_i(c)}$ is the class function of $\widetilde{\Gamma}_n$ which takes value $\pm 2^{-l(\rho)/2} Z_\rho$ on the conjugacy classes $D_\rho^\pm$ and 0 elsewhere. Given $\gamma \in R(\Gamma)$, we denote by $\sigma_n(\gamma)$ the class function on $\widetilde{\Gamma}_n$ which takes value $\pm\frac{n}{\sqrt{2}}\gamma(c)$ on $D_n(c)^\pm, c \in \Gamma_*$, and 0 elsewhere.

The following lemma is not difficult to verify.

Lemma 6.1. *The map ch sends σ_ρ to $a'_{-\rho}$. In particular, it sends $\sigma_n(c)$ to $a_{-n}(c)$ in S_Γ^- and $\sigma_n(\gamma)$ to $a_{-n}(\gamma)$ for $n \in 2\mathbb{Z} + 1$.*

In Sect. 9.2, we will see that the space S_Γ^- has another distinguished basis consisting of generalized Schur Q-functions, which give rise to some integral basis in $R^-(\widetilde{\Gamma}_n)$.

6.2. **The image of $\chi_n(\gamma)$ under ch.** Recall that we have defined a map from $R(\Gamma)$ to $R^-(\widetilde{\Gamma}_n)$ (Subsection 4.3).

Proposition 6.2. *For any $\gamma \in R(\Gamma)$, we have*

$$\sum_{n \geq 0} 2^{\overline{n}/2} ch(\chi_n(\gamma)) z^n = \exp\left(\sum_{n \geq 1, odd} \frac{2}{n} a_{-n}(\gamma) z^n \right), \tag{6.2}$$

where $\overline{n}$ is 0 or 1 according to n is even or odd.

Proof. The character value of $\chi_n(\gamma)$ is given in Corollary 4.5, and we have

$$\begin{aligned}
\sum_{n\geq 0} 2^{\overline{n}/2}\mathrm{ch}(\chi_n(\gamma))z^n &= \sum_{\rho} 2^{l(\rho)} Z_\rho^{-1} \prod_{c\in\Gamma_*} \gamma(c)^{l(\rho(c))} a'_{-\rho(c)} z^{||\rho||} \\
&= \prod_{c\in\Gamma_*}\Big(\sum_{\lambda}(2\zeta_c^{-1}\gamma(c))^{l(\lambda)} z_\lambda^{-1} a_{-\lambda}(c) z^{|\lambda|}\Big) \\
&= \exp\left(\sum_{n\geq 1}\frac{2}{n}\sum_{c\in\Gamma_*}\zeta_c^{-1}\gamma(c)a_{-n}(c)z^n\right) \\
&= \exp\left(\sum_{n\geq 1}\frac{2}{n}\, a_{-n}(\gamma)z^n\right).
\end{aligned}$$

Let β, γ be the characters of two representations of Γ. It follows from (4.15) that

$$\begin{aligned}
&\sum_{n\geq 0} 2^{\overline{n}/2}\mathrm{ch}(\chi_n(\beta-\gamma))z^n \\
=\ &\left(\sum_{n\geq 0} 2^{\overline{n}/2}\mathrm{ch}(\chi_n(\beta))z^n\right)\cdot\left(\sum_{n\geq 0} 2^{\overline{n}/2}\mathrm{ch}(\chi_n(\gamma))(-z)^n\right) \\
=\ &\exp\left(\sum_{n\geq 1, odd}\frac{2}{n}\, a_{-n}(\beta-\gamma)z^n\right).
\end{aligned}$$

Therefore the proposition holds for $\beta-\gamma$, and so for any element $\gamma \in R_{\mathbb{Z}}(\Gamma)$. □

Corollary 6.3. *The formula (4.17) holds for any $\gamma \in R(\Gamma)$. In particular $\chi_n(\xi)$ is self-dual if ξ is self-dual.*

Component-wise, we obtain

$$\mathrm{ch}(\chi_n(\gamma)) = 2^{-\overline{n}/2}\sum_{\rho}\frac{2^{l(\rho)}}{z_\rho}\, a_{-\rho}(\gamma),$$

where the sum runs through all the partitions ρ of n into odd integers.

6.3. **Isometry between R_Γ^- and S_Γ^-.** It is well known that there exists a natural Hopf algebra structure on the symmetric algebra S_Γ^- with the usual multiplication and the comultiplication Δ characterized by

$$\Delta(a_{-n}(\gamma)) = a_{-n}(\gamma)\otimes 1 + 1\otimes a_{-n}(\gamma), \qquad n \in 2\mathbb{Z}_+ + 1. \tag{6.3}$$

Recalling the Hopf algebra structure on R_Γ^- defined in Sect. 3.4, we can easily verify the following proposition as in the untwisted case.

Proposition 6.4. *The characteristic map* $ch : R_\Gamma^- \longrightarrow S_\Gamma^-$ *is an isomorphism of Hopf algebras.*

Proof. By counting dimensions of homogeneous degree subspaces it is easy to see that ch is an isomorphism of vector spaces. The algebra isomorphism follows simply from the Frobenius reciprocity. To check the coalgebra isomorphism we use Proposition 6.2 to pass from the generators $a_n(\gamma)$ to the character $\chi_n(\gamma)$. It is then a simple calculation to verify that $\chi_n(\gamma)$ is group-like under the comultiplication (3.13), and this shows that ch is a Hopf algebra isomorphism by using (6.3). □

Recall that we have defined a bilinear form $\langle\ ,\ \rangle_\xi$ on R_Γ^- and a bilinear form on S_Γ^- denoted by $\langle\ ,\ \rangle'_\xi$. The lemma below follows from our definition of $\langle\ ,\ \rangle'_\xi$ and the comultiplication Δ.

Lemma 6.5. *The bilinear form* $\langle\ ,\ \rangle'_\xi$ *on* S_Γ^- *can be characterized by the following two properties:*

1). $\langle a_{-n}(\beta), a_{-m}(\gamma)\rangle'_\xi = \frac{n}{2}\delta_{n,m}\langle \beta,\gamma\rangle'_\xi, \quad \beta,\gamma \in \Gamma^*, m,n \in 2\mathbb{Z}_+ + 1.$

2). $\langle fg, h\rangle'_\xi = \langle f\otimes g, \Delta h\rangle'_\xi$, *where* $f,g,h \in S_\Gamma^-$, *and the bilinear form on the r.h.s of 2), which is defined on* $S_\Gamma^-\otimes S_\Gamma^-$, *is induced from* $\langle\ ,\ \rangle'_\xi$ *on* S_Γ^-.

Theorem 6.6. *The characteristic map ch is an isometry from the space* $(R_\Gamma^-, \langle\ ,\ \rangle_\xi)$ *to* $(S_\Gamma^-, \langle\ ,\ \rangle'_\xi)$.

Proof. Let f and g be any two super class functions in $R^-(\widetilde{\Gamma}_n)$. By definition of ch (6.1) it follows that

$$\begin{aligned}
&\langle ch(f), ch(g)\rangle'_\xi \\
&= \sum_{\rho,\rho'\in\mathcal{OP}_n(\Gamma_*)} \frac{2^{(l(\rho)+l(\rho'))/2}}{Z_\rho Z_{\rho'}} f(\rho)g(\rho')\langle a'_{-\overline{\rho}}, a'_{-\overline{\rho'}}\rangle'_\xi \\
&= \sum_{\rho,\rho'\in\mathcal{OP}_n(\Gamma_*)} \frac{2^{(l(\rho)+l(\rho'))/2}}{Z_\rho Z_{\rho'}} f(\rho)g(\rho')\frac{Z_{\rho'}}{2^{l(\rho')}}\prod_{c\in\Gamma_*}\xi(c)^{l(\rho(c))}\delta_{\rho,\overline{\rho'}} \\
&= \sum_{\rho\in\mathcal{OP}_n(\Gamma_*)} \frac{1}{Z_\rho} f(\rho)g(\overline{\rho})\prod_{c\in\Gamma_*}\xi(c)^{l(\rho(c))} \\
&= \langle f, g\rangle_\xi,
\end{aligned}$$

where we have used the inner product identity (5.4). □

Remark 6.7. We can also prove it by showing that the characteristic map preserves the inner product of basis elements $\sigma_\rho \in R^-_\Gamma$ and that of $a_{-\rho} = ch(\sigma_\rho) \in S^-_\Gamma$ as in [FJW1].

From now on we will identify the inner product $\langle\ ,\ \rangle_\xi$ on R^-_Γ with the inner product $\langle\ ,\ \rangle'_\xi$ on S^-_Γ. As a special case, the standard Hermitian form on $R^-(\Gamma_n)$ and therefore on R^-_Γ is compatible via the characteristic map ch with the Hermitian form characterized by (5.3) on S^-_Γ.

7. Vertex operators and R^-_Γ

7.1. **A central extension of $R_{\mathbb{Z}}(\Gamma)/2R_{\mathbb{Z}}(\Gamma)$.** From now on we assume that ξ is a self-adjoint virtual character of Γ, and thus $R_{\mathbb{Z}}(\Gamma)$ is an integral lattice under the symmetric bilinear form $\langle\ ,\ \rangle_\xi$.

Let $2R_{\mathbb{Z}}(\Gamma)$ be the sublattice of $R_{\mathbb{Z}}(\Gamma)$ consisting of elements $2\alpha, \alpha \in R_{\mathbb{Z}}(\Gamma)$. The quotient $R_{\mathbb{F}_2}(\Gamma) = R_{\mathbb{Z}}(\Gamma)/2R_{\mathbb{Z}}(\Gamma)$ has an induced abelian group structure and it can also be viewed as an $(r+1)$-dimensional vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. We will denote by $\overline{\alpha}$ the natural image of α in $R_{\mathbb{F}_2}(\Gamma)$. Define c_1 to be the alternating form: $R_{\mathbb{F}_2}(\Gamma)\times R_{\mathbb{F}_2}(\Gamma) \to \mathbb{F}_2$ given by $c_1(\overline{\alpha}, \overline{\beta}) = \langle\alpha,\beta\rangle_\xi + \langle\alpha,\alpha\rangle_\xi\langle\beta,\beta\rangle_\xi\ (mod\, 2)$, and let r_0 be its rank over $\mathbb{F}_2$.

The alternating form c_1 gives rise to a central extension $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$ of the abelian group $R_{\mathbb{F}_2}(\Gamma)$ by the two-element group $\langle\pm 1\rangle$ (see [FLM1]):

$$1 \to <\pm 1> \hookrightarrow \hat{R}^-_{\mathbb{F}_2}(\Gamma) \xrightarrow{\ \breve{}\ } R_{\mathbb{F}_2}(\Gamma) \to 1, \tag{7.1}$$

such that $aba^{-1}b^{-1} = (-1)^{c_1(\breve{a},\breve{b})}$, $a, b \in \hat{R}^-_{\mathbb{F}_2}(\Gamma)$.

The elements of $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$ can be presented as $\pm e_{\overline{\alpha}}$, where $\alpha \in R_{\mathbb{Z}}(\Gamma)$, which implies that $dim(\hat{R}^-_{\mathbb{F}_2}(\Gamma)) = 2^{r+2}$. We note that $e_{\overline{\alpha}} \in \hat{R}^-_{\mathbb{F}_2}(\Gamma)$ satisfies $(e_{\overline{\alpha}})^2 = 1$.

Let Φ be a subgroup of $R_{\mathbb{Z}}(\Gamma)$ which is maximal such that the alternating form c_1 vanishes on $\Phi/2R_{\mathbb{Z}}(\Gamma)$. A variant of the following lemma was given in [FLM2].

Lemma 7.1. *There are $2^{(r+1-r_0)}$ irreducible $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$-module structures on the space $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ such that $-1 \in \hat{R}^-_{\mathbb{F}_2}(\Gamma)$ acts faithfully and*

$$e_{\overline{\alpha}}e_{\overline{\beta}} = e_{\overline{\beta}}e_{\overline{\alpha}}(-1)^{c_1(\overline{\alpha},\overline{\beta})} \tag{7.2}$$

as operators on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$. The dimension of $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ is equal to $2^{\frac{1}{2}r_0}$.

We will denote the elements of $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ by $e^{[\alpha]}$, where $[\alpha] = \alpha + \Phi \in R_{\mathbb{Z}}(\Gamma)/\Phi$. Clearly

$$e^{2[\alpha]} = 1, e^{[\alpha+\beta]} = e^{[\alpha]}e^{[\beta]}.$$

For $\alpha, \beta \in R_{\mathbb{Z}}(\Gamma)$ we write the action of $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$ on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ as

$$e_{\overline{\alpha}}.e^{[\beta]} = \epsilon(\alpha, \beta)e^{[\alpha+\beta]}. \tag{7.3}$$

Then one can check that ϵ is a well-defined cocycle map from $R_{\mathbb{Z}}(\Gamma) \times R_{\mathbb{Z}}(\Gamma) \to \langle \pm 1 \rangle$. One also has $\epsilon(\alpha, \beta) = \epsilon(\alpha, -\beta)$.

7.2. **Twisted Vertex Operators** $X(\gamma, z)$**.** Fix an irreducible $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$-module structure on $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$ described in Eqn. (7.3).

We extend the actions of $e_{\overline{\alpha}}$ to the space of tensor product

$$\mathcal{F}^-_\Gamma = R^-_\Gamma \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi],$$

by letting them act on the R^-_Γ part trivially.

Introduce the operators $H_{\pm n}(\gamma), \gamma \in R(\Gamma), n > 0$ as the following compositions of maps:

$$\begin{aligned}
H_{-n}(\gamma) \quad &: \quad R^-(\widetilde{\Gamma}_m) \stackrel{2^{\overline{n}/2}\chi_n(\gamma)\otimes}{\longrightarrow} R^-(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_m) \stackrel{Ind}{\longrightarrow} R^-(\widetilde{\Gamma}_{n+m}) \\
H_n(\gamma) \quad &: \quad R^-(\widetilde{\Gamma}_m) \stackrel{Res}{\longrightarrow} R^-(\widetilde{\Gamma}_n \hat{\times} \widetilde{\Gamma}_{m-n}) \stackrel{\langle 2^{\overline{n}/2}\chi_n(\gamma), \cdot\rangle_\xi}{\longrightarrow} R^-(\widetilde{\Gamma}_{m-n}).
\end{aligned}$$

Define

$$H_+(\gamma, z) = \sum_{n>0} H_{-n}(\gamma) z^n, \quad H_-(\gamma, z) = \sum_{n>0} H_n(\gamma) z^{-n}.$$

We now define the twisted vertex operators $X_n(\gamma), n \in \mathbb{Z}, \gamma \in R_\Gamma$ by the following generating functions:

$$\begin{aligned}
X^+(\gamma, z) \quad &\equiv \quad X(\gamma, z) \\
&= \quad \sum_{n\in\mathbb{Z}} X_n(\gamma) z^{-n} \\
&= \quad H_+(\gamma, z) H_-(\gamma, -z) e_{\overline{\gamma}}.
\end{aligned} \tag{7.4}$$

We also denote

$$\begin{aligned}
X^-(\gamma, z) \quad &\equiv \quad X(-\gamma, z) = X(\gamma, -z) \\
&= \quad \sum_{n\in\mathbb{Z}} X^-_n(\gamma) z^{-n}.
\end{aligned}$$

The operators $X_n(\gamma)$ are well-defined operators acting on the space $\mathcal{F}^-_\Gamma$. We extend the bilinear form $\langle\ ,\ \rangle_\xi$ on R^-_Γ to $\mathcal{F}^-_\Gamma$ by letting

$$\langle f e^{[\alpha]}, g e^{[\beta]} \rangle_\xi = \langle f, g \rangle_\xi \delta_{[\alpha],[\beta]}, \quad f, g \in R^-_\Gamma, \alpha, \beta \in R_{\mathbb{Z}}(\Gamma).$$

We extend the $\mathbb{Z}_+$-gradation from R^-_Γ to $\mathcal{F}^-_\Gamma$ by letting

$$\deg a_{-n}(\gamma) = n, \quad \deg e_{\overline{\gamma}} = 0.$$

Similarly we extend the bilinear form $\langle\ ,\ \rangle_\xi$ to the space

$$V_\Gamma^- = S_\Gamma^- \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$$

and extend the $\mathbb{Z}_+$-gradation on S_Γ to a $\mathbb{Z}_+$-gradation on V_Γ^-.

The characteristic map ch will be extended to an isometry from $\mathcal{F}_\Gamma^-$ to V_Γ^- by fixing the subspace $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$. We will denote this map again by ch.

7.3. **Twisted Heisenberg algebra and R_Γ^-.** We define $\widetilde{a}_{-n}(\gamma), n \in 2\mathbb{Z}_+ + 1$ to be a map from R_Γ^- to itself by the following composition

$$R^-(\widetilde{\Gamma}_m) \stackrel{\sigma_n(\gamma)\otimes}{\longrightarrow} R^-(\widetilde{\Gamma}_n) \bigotimes R^-(\widetilde{\Gamma}_m) \stackrel{Ind}{\longrightarrow} R^-(\widetilde{\Gamma}_{n+m}).$$

We also define $\widetilde{a}_n(\gamma), n \in 2\mathbb{Z}_+ + 1$ to be a map from R_Γ^- to itself as the composition

$$R^-(\widetilde{\Gamma}_m) \stackrel{Res}{\longrightarrow} R^-(\widetilde{\Gamma}_n) \bigotimes R^-(\widetilde{\Gamma}_{m-n}) \stackrel{\langle\sigma_n(\gamma),\cdot\rangle_\xi}{\longrightarrow} R^-(\widetilde{\Gamma}_{m-n}).$$

We denote by R_Γ^0 the radical of the bilinear form $\langle\ ,\ \rangle_\xi$ in R_Γ^- and denote by $\overline{R}_\Gamma$ the quotient R_Γ^-/R_Γ^0, which inherits the bilinear form $\langle\ ,\ \rangle_\xi$ from R_Γ^-.

Theorem 7.2. *R_Γ^- is a representation of the twisted Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ by letting $a_n(\gamma)$ $(n \in 2\mathbb{Z}+1)$ act as $\widetilde{a}_n(\gamma)$ and C as 1. R_Γ^0 is a subrepresentation of R_Γ^- over $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$ and the quotient $\overline{R}_\Gamma$ is irreducible. The characteristic map ch is an isomorphism of R_Γ^- (resp. R_Γ^0, $\overline{R}_\Gamma$) and S_Γ^- (resp. S_Γ^0, $\overline{S}_\Gamma$) as representations over $\widehat{\mathfrak{h}}_{\Gamma,\xi}[-1]$.*

7.4. **The characteristic map of twisted vertex operators.** We extend the characteristic map ch to a linear map ch: $End(R_\Gamma^-) \rightarrow End(S_\Gamma^-)$ by

$$ch(f).ch(v) = ch(f.v), f \in End(R_\Gamma^-), v \in R_\Gamma^-. \tag{7.5}$$

The relation between the vertex operators defined in (7.4) and the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ is revealed in the following theorem.

Theorem 7.3. *For any $\gamma \in R(\Gamma)$, we have*

$$\begin{aligned} ch\big(H_+(\gamma,z)\big) &= \exp\Bigg(\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_{-n}(\gamma) z^n\Bigg), \\ ch\big(H_-(\gamma,z)\big) &= \exp\Bigg(\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_n(\gamma) z^{-n}\Bigg). \end{aligned}$$

Proof. Observe that the operator $H_+(\gamma, z)$ is the adjoint operator of $H_-(\gamma, z^{-1})$ with respect to the bilinear form $\langle\ ,\ \rangle_\xi$. Then the theorem follows from Lemma 6.1 and Proposition 6.2 by invoking the characteristic map. $\square$

As a consequence we have

$$\begin{aligned} & \mathrm{ch}\big(X(\gamma, z)\big) \\ = \;& \exp\Bigg(\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_{-n}(\gamma) z^n\Bigg) \exp\Bigg(-\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_n(\gamma) z^{-n}\Bigg) e_{\overline{\gamma}}. \end{aligned}$$

Thus the characteristic map identifies the twisted vertex operators $X(\gamma, z)$ defined via finite groups $\widetilde{\Gamma}_n$ with the usual twisted vertex operators of [FLM1, FLM2].

8. Vertex representations and the McKay correspondence

8.1. **Product of two vertex operators.** The normal ordered product $: X(\alpha, z)X(\beta, w) :$, $\alpha, \beta \in R(\Gamma)$ of two vertex operators is defined as follows:

$$: X(\alpha, z)X(\beta, w) := H_+(\alpha, z)H_+(\beta, w)H_-(\alpha, -z)H_-(\beta, -w)e_{\overline{\alpha}+\overline{\beta}}.$$

In the following theorem and later the expression $\left(\frac{z-w}{z+w}\right)^{\langle \alpha,\beta\rangle_\xi}$ represents the power series expansion in the variable $\frac{w}{z}$.

Theorem 8.1. *For $\alpha, \beta \in R(\Gamma)$ one has the following operator product expansion identity for twisted vertex operators.*

$$X(\alpha, z)X(\beta, w) \;=\; \epsilon(\alpha, \beta) : X(\alpha, z)X(\beta, w) : \left(\frac{z-w}{z+w}\right)^{\langle \alpha,\beta\rangle_\xi}.$$

Proof. It suffices to compute that

$$\begin{aligned} & ch(H_-(\alpha, -z)H_+(\beta, w)) \\ = \;& \exp\Bigg(-\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_n(\alpha) z^{-n}\Bigg) \exp\Bigg(\sum_{n\geq 1,\, odd} \frac{2}{n}\, a_{-n}(\beta) w^n\Bigg) \\ = \;& ch(H_+(\beta, w)H_-(\alpha, -z)) \exp\Bigg(-\langle \alpha, \beta\rangle_\xi \sum_{n\geq 1,\, odd} \frac{2}{n}\, z^{-n} w^n\Bigg) \\ = \;& ch(H_+(\beta, w)H_-(\alpha, -z)) \left(\frac{z-w}{z+w}\right)^{\langle \alpha,\beta\rangle_\xi}. \end{aligned}$$

$\square$

The following proposition is easy to check.

Proposition 8.2. *Given* $\alpha \in R(\Gamma), \beta \in R_{\mathbb{Z}}(\Gamma)$ *and* $n \in 2\mathbb{Z}+1$, *we have*

$$[a_n(\alpha), X(\beta, z)] = \langle \alpha, \beta \rangle_\xi X(\beta, z) z^n.$$

8.2. **Twisted affine Lie algebra $\widehat{\mathfrak{g}}[-1]$ and twisted toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}[-1]$.** Let $\mathfrak{g}$ be a rank r complex simple Lie algebra of ADE type, and let $\overline{\Delta}$ be the root system generated by a set of simple roots $\alpha_1 . \dots , \alpha_r$. Let α_{max} be the highest root. The Lie algebra is generated by the Chevalley generators $e_{\alpha_i}, e_{-\alpha_i}, h_i = h_{\alpha_i}$. We normalize the invariant bilinear form on $\mathfrak{g}$ by $(\alpha_{max}, \alpha_{max}) = 2$.

Let θ be an automorphism of $\mathfrak{g}$ of order k and let $\omega = exp(2\pi i/k)$. The automorphism θ induces a $\mathbb{Z}/k\mathbb{Z}$-gradation for $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{g}_i, \qquad \mathfrak{g}_i = \{g \in \mathfrak{g} | \theta(g) = \omega^i g\},$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

The twisted affine Lie algebra $\widehat{\mathfrak{g}}[\theta]$ is the graded vector space

$$\widehat{\mathfrak{g}}[\theta] = \bigoplus_{i=0}^{k-1} \mathfrak{g}_i \otimes t^i \mathbb{C}[t^k, t^{-k}] \bigoplus \mathbb{C}C \tag{8.1}$$

with the commutating relations

$$[a(n), b(m)] = [a, b](n+m) + \frac{n}{k}\delta_{n,-m}(a|b)C, \tag{8.2}$$

$$[C, a(n)] = 0, \quad a, b \in \mathfrak{g}, n, m \in \mathbb{Z}, \tag{8.3}$$

where we used the notation

$$a(n) = a \otimes t^n, \quad a \in \mathfrak{g}, n \in \mathbb{Z}.$$

When θ is the identity, $\widehat{\mathfrak{g}}[1]$ becomes the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$. Let $A = (a_{ij})_{0 \le i,j \le r}$ be the affine Cartan matrix associated to $\widehat{\mathfrak{g}}$. The submatrix $(a_{ij})_{1 \le i,j \le r}$ is the Cartan matrix of $\mathfrak{g}$.

The linear map

$$e_{\alpha_i} \longrightarrow e_{-\alpha_i}$$
$$h_{\alpha_i} \longrightarrow -h_{\alpha_i}$$

defines an involution of the Lie algebra $\mathfrak{g}$. We will denote the associated twisted affine Lie algebra by $\widehat{\mathfrak{g}}[-1]$. Let $k_\alpha = \frac{1}{2}(e_\alpha + e_{-\alpha})$ and $p_\alpha =$

$\frac{1}{2}(e_\alpha - e_{-\alpha})$, where $\alpha \in \overline{\Delta}$. It is easily seen that

$$\mathfrak{g}_0 = \bigoplus \mathbb{C}k_\alpha,$$
$$\mathfrak{g}_1 = \bigoplus \mathbb{C}p_\alpha \oplus \bigoplus \mathbb{C}h_\alpha$$

The *basic twisted representation* V of $\widehat{\mathfrak{g}}[-1]$ is the irreducible highest weight representation generated by a highest weight vector which is annihilated by $a(n), n \in \mathbb{Z}_+, a \in \mathfrak{g}$ and C acts on V as the identity operator.

We now introduce the complex twisted toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}[-1]$ (associated to $\mathfrak{g}$) with the following presentation: the generators are

$$C, h_i(m), x_n(\pm\alpha_i), m \in 2\mathbb{Z}+1, n \in \mathbb{Z}, i = 0, \ldots, r;$$

and the relations are given by: C is central, and

$$\begin{aligned}
&x_n(\alpha_i) = (-1)^n x_n(-\alpha_i),\\
&[h_i(m), h_j(m')] = \frac{m}{2} a_{ij}\delta_{m,-m'}C,\\
&[h_i(n), x_m(\alpha_j)] = a_{ij}x_{n+m}(\alpha_j),\\
&[x_n(\alpha_i), x_{n'}(-\alpha_i)] = 8\{h_i(n+n') + n\delta_{n,-n'}C\},\\
&\sum_{s=0}^{a_{ij}} \binom{a_{ij}}{s} [x_{n+s}(\alpha_i), x_{n'-a_{ij}-s}(\alpha_j)] = 0, \quad \text{if } a_{ij} \geq 0\\
&\sum_{s=0}^{-a_{ij}} (-1)^s \binom{-a_{ij}}{s} [x_{n+s}(\alpha_i), x_{n'-a_{ij}-s}(\alpha_j)] = 0, \quad \text{if } a_{ij} < 0
\end{aligned} \tag{8.4}$$

where $n, n' \in \mathbb{Z}$, $m, m' \in 2\mathbb{Z}+1$, $i, j = 0, 1, \ldots, r$, and $h_i(2n) = 0$ for $n \in \mathbb{Z}$. The twisted toroidal algebra is the $q \to 1$ limit of the twisted quantum current algebra [J2] (see a slightly different form in [DI]).

Set $k_i(2n) = \frac{1}{4}x_{2n}(\alpha_i)$ and $p_i(2n+1) = \frac{1}{4}x_{2n+1}(\alpha_i)$. One can check that the relations given in (8.4) are consequences of the twisted algebra $\widehat{\mathfrak{g}}[-1]$ for the case of $\theta = -1$ defined in (8.2) (cf. the proof of Theorem 8.3 later) .

8.3. **Realization of twisted vertex representations.** Let Γ be a finite subgroup of $SL_2(\mathbb{C})$ and the virtual character ξ to be twice the trivial character minus the character of the two-dimensional defining representation of $\Gamma \hookrightarrow SL_2(\mathbb{C})$. It is known [Mc] that the matrix $A = (a_{ij})_{0\leq i,j\leq r}$ in Sect. 4.4 is the Cartan matrix for the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$.

The following theorem provides a finite group realization of the vertex representation of the twisted toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}[-1]$ on $\mathcal{F}_\Gamma^-$.

Theorem 8.3. *A vertex representation of the twisted toroidal Lie algebra* $\widehat{\widehat{\mathfrak{g}}}[-1]$ *is defined on the space* $\mathcal{F}_\Gamma^-$ *by letting*

$$\begin{aligned} x_n(\alpha_i) &\mapsto X_n(\gamma_i), \qquad x_n(-\alpha_i) \mapsto \epsilon(\gamma_i,\gamma_i)X_n(-\gamma_i),\\ h_i(m) &\mapsto a_m(\gamma_i), \qquad C \mapsto 1, \end{aligned}$$

where $n \in \mathbb{Z}, m \in 2\mathbb{Z}+1, 0 \le i \le r$.

Proof. All the commutation relations without binomial coefficients are easy consequences of Proposition 8.2 and Theorem 8.1 by the usual vertex operator calculus in the twisted picture (see [FLM2, J1]). The corresponding relations with binomial coefficients in V_Γ^- are equivalent to

$$\begin{aligned} (z+w)^{a_{ij}}[X(\gamma_i,z),X(\gamma_j,w)] &= 0, \qquad a_{ij} \ge 0,\\ (z-w)^{-a_{ij}}[X(\gamma_i,z),X(\gamma_j,w)] &= 0, \qquad a_{ij} < 0. \end{aligned}$$

This is again proved by using Theorem 8.1 with the same method as in the quantum vertex operators [J2]. □

Recall that $\delta = \sum_{i=0}^r d_i\gamma_i$ generates the one-dimensional radical $R_\mathbb{Z}^0$ of the bilinear form $\langle\ ,\ \rangle_\xi$ in $R_\mathbb{Z}(\Gamma)$, where d_i is the degree of the irreducible character γ_i of Γ. The lattice $R_\mathbb{Z}(\Gamma)$ in this case can be identified with the root lattice for the corresponding affine Lie algebra. The quotient lattice $R_\mathbb{Z}(\Gamma)/R_\mathbb{Z}^0$ inherits a positive definite integral bilinear form. Denote by $\overline{\Gamma}^*$ the set of non-trivial irreducible characters of Γ:

$$\overline{\Gamma}^* = \{\gamma_1,\gamma_2,\dots,\gamma_r\}.$$

Let $R_\mathbb{Z}(\overline{\Gamma}^*)$ be the sublattice of $R_\mathbb{Z}(\Gamma)$ generated by $\overline{\Gamma}^*$. Denote by $Sym(\overline{\Gamma}^*)$ the symmetric algebra generated by $a_{-n}(\gamma_i)$, $n \in 2\mathbb{Z}_+ +1$, $i = 1,\dots,r$. Equipped with the bilinear form $\langle\ ,\ \rangle_\xi$, $Sym(\overline{\Gamma}^*)$ is isometric to $\overline{S}_\Gamma$ which is in turn isometric to $\overline{R}_\Gamma$ as well. The irreducible $\hat{R}_{\mathbb{F}_2}^-(\Gamma)$-module $\mathbb{C}[R_\mathbb{Z}(\Gamma)/\Phi]$ induces an irreducible $\hat{R}_{\mathbb{F}_2}^-(\overline{\Gamma}^*)$-module structure on $\mathbb{C}[R_\mathbb{Z}(\overline{\Gamma}^*)/\overline{\Phi}]$ given by the restriction of the alternating form c_1. We let $\overline{r}_0$ denote the rank of the restriction of c_1, then the statement of Lemma 7.1 also holds for the sublattice $R_{\mathbb{F}_2}(\overline{\Gamma}^*)$ and $\hat{R}_{\mathbb{F}_2}^-(\overline{\Gamma}^*)$. In this case if the determinant of the Cartan matrix is an odd integer, then $\overline{r}_0 = 0$ and the space $\mathbb{C}[R_\mathbb{Z}(\overline{\Gamma}^*)/\overline{\Phi}]$ is trivial.

We define

$$\begin{aligned} \overline{V}_\Gamma &= \overline{S}_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\overline{\Gamma}^*)/\overline{\Phi}] \cong Sym(\overline{\Gamma}^*) \bigotimes \mathbb{C}[R_\mathbb{Z}(\overline{\Gamma}^*)/\overline{\Phi}],\\ \overline{\mathcal{F}}_\Gamma &= \overline{R}_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\overline{\Gamma}^*)/\overline{\Phi}]. \end{aligned}$$

Obviously ch, when restricted to $\overline{\mathcal{F}}_\Gamma$, is an isometric isomorphism onto $\overline{V}_\Gamma$.

The space $\mathcal{F}_\Gamma^-$ associated to the lattice $R_\mathbb{Z}(\Gamma)$ is isomorphic to the tensor product of the space $\overline{\mathcal{F}}_\Gamma$ associated to $\overline{R}_\mathbb{Z}(\Gamma)$ and the space associated to the rank 1 lattice $\mathbb{Z}\delta$ equipped with the zero bilinear form.

The identity for a product of vertex operators $X(\gamma, z)$ associated to $\gamma \in \overline{\Delta}$ (cf. Theorem 8.3) implies that $\overline{V}_\Gamma$ provides a realization of the vertex representation of $\widehat{\mathfrak{g}}[-1]$ on $\overline{V}_\Gamma$ (cf. [FLM1]). The following theorem establishes a direct link from the finite group $\Gamma \in SL_2(\mathbb{C})$ to the affine Lie algebra $\widehat{\mathfrak{g}}[-1]$. This gives a twisted version of the new form of the McKay correspondence given in [FJW1].

Theorem 8.4. *The operators $X_n(\gamma), \gamma \in \overline{\Delta}, a_n(\gamma_i), i = 1, 2, \ldots, r, n \in \mathbb{Z}$ define an irreducible representation of the affine Lie algebra $\widehat{\mathfrak{g}}[-1]$ on $\overline{\mathcal{F}}_\Gamma$ isomorphic to the twisted basic representation.*

9. Vertex operators and irreducible characters of $\widetilde{\Gamma}_n$

In this section we specialize ξ to be the trivial character γ_0 of Γ. We will describe how to obtain the character table for the spin supermodules of $\widetilde{\Gamma}_n$ from our vertex operator approach, generalizing [J1].

9.1. **Algebra of vertex operators for $\xi = \gamma_0$.** In this case the weighted bilinear form reduces to the standard one $\langle\ ,\ \rangle$ and $R_\mathbb{Z}(\Gamma)$ is isomorphic to the lattice $\mathbb{Z}^{r+1}$ with the standard integral bilinear form. Recall that $\langle \gamma_i, \gamma_j \rangle = \delta_{ij}$. For simplicity we will only consider the vertex representations on the space R_Γ. In the following result the bracket $\{\ ,\ \}$ denotes the anti-commutator.

Theorem 9.1. *The operators $X_n^+(\gamma_i), X_n^-(\gamma_i)$ $(n \in \mathbb{Z}, 0 \le i \le r)$ generate a generalized Clifford algebra:*

$$\begin{aligned}
[X_n^+(\gamma_i), X_{n'}^+(\gamma_j)] &= 0, \quad i \neq j \\
[X_n^-(\gamma_i), X_{n'}^-(\gamma_j)] &= 0, \quad i \neq j \\
\{X_n^+(\gamma_i), X_{n'}^+(\gamma_i)\} &= 2(-1)^n \delta_{n,-n'}, \\
\{X_n^-(\gamma_i), X_{n'}^-(\gamma_i)\} &= 2(-1)^n \delta_{n,-n'}, \\
[X_n^+(\gamma_i), X_{n'}^-(\gamma_j)] &= 0, \quad i \neq j \\
\{X_n^+(\gamma_i), X_{n'}^-(\gamma_i)\} &= 2\delta_{n,-n'}.
\end{aligned} \tag{9.1}$$

Proof. It follows from the standard vertex operator calculus (cf. [J1]) by using Theorem 8.1. ∎

Therefore we see that R^-_Γ is isomorphic to the tensor product of $r+1$ copies of the space R^-, the sum of Grothendieck groups of spin characters of $\widetilde{S}_n$-supermodules.

Remark 9.2. Let $A = (1-\delta_{ij})_{(r+1)\times(r+1)}$, the matrix of the alternating form c_1 over $\mathbb{F}_2$, then $A^2 = \overline{r}I$. Here $\overline{r} = 0$ if r is even and 1 if r is odd. Consequently it follows from Lemma 7.1 there are exactly $2^{\overline{r+1}}$ irreducible $\hat{R}^-_{\mathbb{F}_2}(\Gamma)$-module structures on the $2^{\lceil\frac{r+1}{2}\rceil}$-dimensional space $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)/\Phi]$. One of the (at most) two irreducible module structures is given by the cocycle $\epsilon(\gamma_i,\gamma_j) = 1$, for $i \le j$, and $\epsilon(\gamma_i,\gamma_j) = -1$, for $i > j$. Then the vertex operators $X^\pm_n(\gamma_i)$ generate the twisted Clifford algebra on the space $\mathcal{F}^-_\Gamma$ defined by $\{X^\pm_n(\gamma_i), X^\pm_{n'}(\gamma_i)\} = 2(-1)^n\delta_{ij}\delta_{n,-n'}$ and $\{X^+_n(\gamma_i), X^-_{n'}(\gamma_i)\} = 2\delta_{ij}\delta_{n,-n'}$.

9.2. **Super spin character tables of $\widetilde{\Gamma}_n$ and vertex operators.** We now use the twisted vertex operators to construct all irreducible characters of spin supermodules of $\widetilde{\Gamma}_n$ for all n.

Let $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)$ be the lattice generated by the characters of spin irreducible $\widetilde{\Gamma}_n$-supermodules. Then $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)\otimes\mathbb{C} \simeq R^-(\widetilde{\Gamma}_n)$.

First we construct a special orthonormal basis in R^-_Γ and then use them to give irreducible characters of spin $\widetilde{\Gamma}_n$-supermodules. The vertex operator $X_n(\gamma)$ is defined as in (7.4) except that we drop $e_{\overline{\gamma}}$. The following is easily seen (cf. [J1]).

Lemma 9.3. *For $n\in\mathbb{Z}$, $\alpha\in R(\Gamma)$ and $\gamma\in\Gamma^*$, we have*

$$X_{-n}(\pm\gamma).1 = \delta_{n,0}, \qquad n \ge 0.$$

For a m-tuple index $\phi = (\phi_1,\cdots,\phi_m)\in\mathbb{Z}^m$ we denote

$$X_\phi(\gamma) = X_{\phi_1}(\gamma)\cdots X_{\phi_m}(\gamma).1,$$
$$x_\phi(\gamma) = (X_{\phi_1}(\gamma).1)\cdots(X_{\phi_m}(\gamma).1).$$

We also define the raising operator R_{ij} by

$$R_{ij}(\phi_1,\cdots,\phi_m) = (\phi_1,\cdots,\phi_i+1,\cdots,\phi_j-1,\cdots,\phi_m).$$

Then we define the action of the raising operator R_{ij} on $X_\phi(\gamma)$ or $x_\phi(\gamma)$ by $X_{R_{ij}\phi}(\gamma)$ or $x_{R_{ij}\phi}(\gamma)$.

Given $\lambda\in\mathcal{OP}(\Gamma^*)$, we define

$$X_\lambda = \prod_{\gamma\in\Gamma^*} X_{-\lambda(\gamma)}(\gamma).$$

Similarly we define $x_\lambda = \prod_{\gamma\in\Gamma^*} x_{\lambda(\gamma)}(\gamma)$.

Theorem 9.4. *The vectors X_λ for $\lambda = (\lambda(\gamma))_{\gamma\in\Gamma^*} \in \mathcal{SP}(\Gamma^*)$ form an orthogonal basis in the vector space R_Γ^- with $\langle X_\lambda, X_\mu\rangle = 2^{l(\lambda)}\delta_{\lambda,\mu}$. Moreover, we have that*

$$X_\lambda = \prod_{\gamma\in\Gamma_*}\prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}X_\lambda(\gamma) \tag{9.2}$$

$$= x_\lambda + \sum_{\lambda\gg\mu} c_{\lambda,\mu}x_\mu, \tag{9.3}$$

where $c_{\lambda,\mu}\in\mathbb{Z}$, and R_{ij} is the raising operator.

Proof. The generalized Clifford algebra structure (9.1) implies that the nonzero elements

$$\prod_{\gamma\in\Gamma^*} X_{-n_1}(\gamma)\cdots X_{-n_l}(\gamma).1$$

of distinct indices generate a spanning set in the space R_Γ^-. To see that they satisfy the raising operator expansion we compute that

$$\begin{aligned}&X(\gamma,z_1)\cdots X(\gamma,z_l).1\\ &=: X(\gamma,z_1)\cdots X(\gamma,z_l) : \prod_{i<j}\frac{z_i-z_j}{z_i+z_j}.\end{aligned}$$

Using the result in [J1] it follows that this is exactly the generating function of the raising operator expansion at the case Γ is trivial under the isomorphism ch. In other words, equation (9.2) is true when λ is a characteristic partition-valued function.

Next the orthogonality follows from the generalized Clifford algebra commutation relations in Theorem 9.1. The orthogonality relations show that the raising operator is not affected by the character γ, hence the general case follows by multiplying the raising operator formula for each γ. □

The corresponding basis in S_Γ^- are the classical symmetric functions called Schur's Q-function [J1]. For any strict partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, the Schur's Q-function Q_λ is determined by [S, M]

$$Q_\lambda(y_1,\ldots,y_n) = 2^l\sum_{\sigma\in S_n/S_{n-l}} y_{\sigma(1)}^{\lambda_1}\cdots y_{\sigma(l)}^{\lambda_l}\prod_{\lambda_i>\lambda_j}\frac{y_{\sigma(i)}+y_{\sigma(j)}}{y_{\sigma(i)}-y_{\sigma(j)}}, \tag{9.4}$$

where S_{n-l} acts on $y_{l+1},\cdots,y_n$ and we allow $\lambda_j = 0$ for $j = l+1,\cdots,n$. It is known (see for example [M, J1]) that Q_λ, $\lambda\in\mathcal{SP}_n$ form a basis in the subring of symmetric functions generated by the power sums $p_1, p_3, p_5, \ldots$.

We can think of $a_{-n}(\gamma), n > 0, \gamma\in\Gamma^*$ as the n-th power sum in a sequence of variables $y_\gamma = (y_{i\gamma})_{i\geq 1}$. In this way we identify the space

S_Γ^- with a distinguished subspace of symmetric functions generated by odd degree power sums indexed by Γ^*. In particular given a strict partition λ we denote by $Q_\lambda(\gamma)$ the Schur's Q-function associated to y_γ. We also denote by $Q_\lambda(\gamma)$ the corresponding element in S_Γ^- by the identification of S_Γ^- and R_Γ^-. For $\lambda \in \mathcal{P}(\Gamma^*)$, we denote

$$Q_\lambda = \prod_{\gamma\in\Gamma^*} Q_{\lambda(\gamma)}(\gamma) \in S_\Gamma^-. \tag{9.5}$$

For $\lambda \in \mathcal{SP}(\Gamma^*)$ we define

$$\overline{Q}_\lambda = 2^{-(l(\lambda)-d(\lambda))/2} Q_\lambda, \tag{9.6}$$

where $d(\lambda)$ is the parity of λ (see (2.14). Similarly we define $\overline{q}_\lambda = 2^{-(l(\lambda)-d(\lambda))/2}\prod_{\gamma\in\Gamma^*,i} q_{\lambda_i(\gamma)}(\gamma)$, where $q_m(\gamma) = Q_{(m)}(\gamma)$. In particular we have

$$ch(\chi_n(\gamma)) = \overline{q}_n = 2^{-\overline{n}/2} q_n(\gamma) = 2^{-\overline{n}/2}\overline{Q}_{(n)}(\gamma).$$

The following result immediately follows from Theorem 9.4 combined with the characteristic map ch and (9.6).

Proposition 9.5. *For strict partition-valued function $\lambda \in \mathcal{SP}(\Gamma^*)$ we have*

$$\begin{aligned} &\langle \overline{Q}_\lambda, \overline{Q}_\mu\rangle = 0 \qquad \text{if } \lambda \neq \mu, \\ &\langle \overline{Q}_\lambda, \overline{Q}_\lambda\rangle = \begin{cases} 1 & \text{if } \lambda \text{ is even,} \\ 2 & \text{if } \lambda \text{ is odd.}\end{cases} \end{aligned} \tag{9.7}$$

The following result generalizes a similar result of [Jo] for trivial Γ.

Lemma 9.6. *Under the characteristic map ch the symmetric functions $\overline{q}_\lambda$ corresponds to a character in $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)$ and $\overline{Q}_\lambda$ correspond to a virtual character in $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)$.*

Proof. Observe that the tensor product of two irreducible supermodules of type Q is a sum of two irreducible supermodule of type M (see Proposition 3.7). Computing its inner product we know that for positive odd integers m and n the character $\chi_n(\gamma)\chi_m(\gamma)$ is twice of some irreducible character. Then by induction we see that $ch^{-1}(\overline{q}_\lambda)$ is a character in $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)$. From Theorem 9.4 we see that $\overline{Q}_\lambda$ is a $\mathbb{Z}$-linear combination of $\overline{q}_\mu$ with $\lambda \gg \mu$, hence $\overline{Q}_\lambda$ is a virtual character of $\widetilde{\Gamma}_n$. $\square$

For $\lambda \in \mathcal{SP}_n(\Gamma^*)$, we define $\widetilde{\Gamma}_\lambda = \widetilde{\Gamma}_{\lambda(\gamma_0)} \hat{\times} \cdots \hat{\times} \widetilde{\Gamma}_{\lambda(\gamma_r)}$. For a partition μ and an irreducible character γ of Γ, we define the spin character $\chi_\mu(\gamma)$ of $\widetilde{\Gamma}_\mu$ to be $\chi_{\mu_1}(\gamma)\otimes\cdots\otimes\chi_{\mu_l}(\gamma)$ (see Corollary 4.5).

Theorem 9.7. *For each strict partition-valued function $\lambda \in \mathcal{SP}_n(\Gamma^*)$, the vector $\overline{Q}_\lambda$ corresponds, under the characteristic map ch, to the irreducible character χ_λ of the spin $\widetilde{\Gamma}_n$-supermodule given by a $\mathbb{Z}$-linear combination of*

$$Ind_{\widetilde{\Gamma}_\rho}^{\widetilde{\Gamma}_n} \chi_{\rho(\gamma_0)}(\gamma_0) \otimes \cdots \otimes \chi_{\rho(\gamma_r)}(\gamma_r), \tag{9.8}$$

where $\rho \leqslant \lambda$ and the first summand is $\rho = \lambda$ with multiplicity one. The parity of χ_λ is equal to $d(\lambda) = n - l(\lambda) \, (mod\, 2)$. Its character at the conjugacy class of type $\mu \in \mathcal{OP}_n(\Gamma_*)$ is equal to the matrix coefficient

$$2^{(l(\mu)-l(\lambda)+d(\lambda))/2} \langle X_\lambda, a_{-\mu} \rangle. \tag{9.9}$$

Moreover the degree of the character is equal to

$$2^{\lfloor (n-l(\lambda))/2 \rfloor} n! \prod_{\gamma \in \Gamma^*} \Big(\frac{deg(\gamma)^{|\lambda(\gamma)|}}{\prod_{1 \le i \le l(\lambda(\gamma))} \lambda_i(\gamma)!} \prod_{i<j} \frac{\lambda_i(\gamma) - \lambda_j(\gamma)}{\lambda_i(\gamma) + \lambda_j(\gamma)} \Big), \tag{9.10}$$

where $\lfloor a \rfloor$ denotes the smallest integer $\geq a$.

Proof. Suppose we know that $\overline{Q}_\lambda$ corresponds to the character of an irreducible $\widetilde{\Gamma}_n$-supermodule. By (5.4) and definition of the characteristic map we see immediately that the linear combination (9.8) is given by the vertex operator structure and the matrix coefficient (9.9) give the character table of all irreducible supermodules.

First we observe that the number of irreducible spin supermodules of type M is equal to the number of even strict partition-valued functions, which are realized by the vectors $2^{-(l(\lambda)-d(\lambda))/2} X_\lambda(\gamma)$ ($\lambda \in \mathcal{SP}^0(\Gamma^*)$) up to signs. As for the vectors $2^{-(l(\lambda)-d(\lambda))/2} X_\lambda(\gamma)$ with ($\lambda \in \mathcal{SP}^1(\Gamma^*)$), it follows from Theorem 9.4 and Proposition 3.5 that each of such vectors corresponds to a virtual irreducible character in $R^-_{\mathbb{Z}}(\widetilde{\Gamma}_n)$ of type Q, since the case of sum or difference of two irreducible characters of type M is ruled out by the orthogonality. To show that they correspond to actually irreducible characters it is sufficient to show that the value of

$$ch^{-1}(2^{-(l(\lambda)-d(\lambda))/2} X_\lambda(\gamma))$$

at the conjugacy class of the identity element of $\widetilde{\Gamma}_n$ is positive.

Let $c^0 \in \Gamma_*$ be the class consisting of the identity in Γ. The type of the identity element in $\widetilde{\Gamma}_n$ is the partition-valued function ρ such that $\rho(c^0) = (1^n)$ and $\rho(c) = 0$ for $c \neq c^0$. Recall from (5.2) that $a_m(c^0) = \sum_{\gamma \in \Gamma^*} deg(\gamma) a_m(\gamma)$. By comparing weights and using orthogonality

(Theorem 9.4) we have

$$\begin{aligned}\langle X_\lambda, a'_{-\rho}\rangle &= \langle X_{\lambda(\gamma)}, a_{-1}^n(c^0)\rangle\\ &= \langle \prod_{\gamma\in\Gamma^*} X_{\lambda(\gamma)}(\gamma), \big(\sum_{\gamma\in\Gamma^*}(deg\gamma)a_{-1}(\gamma)\big)^n\rangle\\ &= n!\prod_{\gamma\in\Gamma^*}\frac{(deg\gamma)^{|\lambda(\gamma)|}}{|\lambda(\gamma)|!}\prod_{\gamma\in\Gamma^*}\langle X_{\lambda(\gamma)}(\gamma), a_{-1}^{|\lambda(\gamma)|}(\gamma)\rangle.\end{aligned}$$

By the result of (6.51) in [J1] we have

$$\langle X_{\lambda(\gamma)}(\gamma), a_{-1}^{|\lambda(\gamma)|}(\gamma)\rangle = \frac{|\lambda(\gamma)|!}{\lambda_1(\gamma)!\cdots\lambda_{l(\lambda(\gamma))}(\gamma)!}\prod_{i<j}\frac{\lambda_i(\gamma)-\lambda_j(\gamma)}{\lambda_i(\gamma)+\lambda_j(\gamma)},$$

which implies the formula (9.10), thus the theorem is proved. □

The irreducible spin $\widetilde{\Gamma}_n$-supermodules can be described easily as follows. For each irreducible character $\gamma\in\Gamma^*$ let U_γ be the irreducible Γ-module affording γ. For each strict partition ν let V_ν be the corresponding irreducible spin supermodule of $\widetilde{S}_n$. Using the construction of Sect. 4.3, we see that $U_\gamma^{\otimes n}\otimes V_\nu$ is a spin $\widetilde{\Gamma}_n$-supermodule.

Proposition 9.8. *For each strict partition-valued function $\lambda=(\lambda(\gamma))\in\mathcal{SP}_n(\Gamma^*)$, with m of the partitions $\lambda(\gamma)$ being odd, the super tensor product*

$$\prod_{\gamma\in\Gamma^*}\big(U_\gamma^{\otimes l(\lambda(\gamma))}\otimes V_{\lambda(\gamma)}\big)$$

decomposes completely into $2^{\lceil m/2\rceil}$ copies of an irreducible spin $\widetilde{\Gamma}_\lambda$-supermodule. Denote this irreducible module by W_λ. Then the induced supermodule $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n}W_\lambda$ is the irreducible spin $\widetilde{\Gamma}_n$-supermodule corresponding to λ, and it is of type M or Q according to $d(\lambda)=n-l(\lambda)$ is even or odd.

Proof. Let V_λ be the irreducible spin $\widetilde{\Gamma}_n$-supermodule corresponding to λ. It follows from Theorem 9.7 that V_λ is an irreducible component of $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n}W_\lambda$.

Note that the supermodule $U_\gamma^{\otimes l(\lambda(\gamma))}\otimes V_{\lambda(\gamma)}$ is irreducible and is of type M (or type Q) according to $d(\lambda(\gamma))=|\lambda(\gamma)|-l(\lambda(\gamma))$ even (or odd). Let $\lambda(\gamma_{i_0}),\cdots,\lambda(\gamma_{i_{m-1}})$ be odd partitions, and let $\lambda(\gamma_{i_m}),\cdots,\lambda(\gamma_{i_r})$ be even partitions. Then the parity of λ equals the parity of m, i.e., $d(\lambda)=n-l(\lambda)\equiv m\,(mod\,2)$.

It follows from Proposition 3.7 that $\prod_{\gamma\in\Gamma^*}U_\gamma^{\otimes l(\lambda(\gamma))}\otimes V_{\lambda(\gamma)}$ decomposes completely into $2^{\lceil m/2\rceil}$ copies of the irreducible supermodule W_λ

of $\widetilde{\Gamma}_\lambda$, and W_λ is of type M if m is even and of type Q otherwise. The degree of $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n} W_\lambda$ equals to $\frac{|\widetilde{\Gamma}_n|}{|\widetilde{\Gamma}_\lambda|} deg(W_\lambda)$, and we have

$$\begin{aligned} deg(W_\lambda) &= 2^{-\lceil m/2 \rceil} \prod_{\gamma \in \Gamma^*} deg(\gamma)^{l(\lambda(\gamma))} deg(V_{\lambda(\gamma)}) \\ &= 2^{-\lceil m/2 \rceil} \prod_{\gamma \in \Gamma^*} \left(2^{\lfloor \frac{d(\lambda(\gamma))}{2} \rfloor} \frac{deg(\gamma)^{l(\lambda(\gamma))} |\lambda(\gamma))|!}{\prod_{1 \le i \le l(\lambda(\gamma))} \lambda_i(\gamma)!} \prod_{i<j} \frac{\lambda_i(\gamma) - \lambda_j(\gamma)}{\lambda_i(\gamma) + \lambda_j(\gamma)}\right), \end{aligned}$$

where we have used the degree formula (9.10) for the special case of $\widetilde{S}_n$. The exponents of 2 in the first factor and the product sum up to

$$\begin{aligned} &\sum_{\gamma \in \Gamma^*} \lfloor d(\lambda(\gamma)) \rfloor - \lceil m/2 \rceil \\ &= \sum_{s=0}^{m-1} \frac{d(\lambda(\gamma_{i_s})) + 1}{2} + \sum_{s=m}^{r} \frac{d(\lambda(\gamma_{i_s}))}{2} - \frac{m - \overline{m}}{2} \\ &= \frac{n - l(\lambda)}{2} + \frac{\overline{m}}{2} = \lfloor \frac{n - l(\lambda)}{2} \rfloor, \end{aligned}$$

where $\overline{m} = 0$ or 1 according to m is even or odd. Thus the degree of $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n} W_\lambda$ is exactly the one given by Eqn. (9.10). Hence $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n} W_\lambda$ is the irreducible spin $\widetilde{\Gamma}_n$-supermodule V_λ corresponding to λ. □

References

[DI] J. Ding and K. Iohara, *Generalizations and deformations of Drinfeld quantum affine algebras*, Lett. Math. Phys. **41** (1997), 181-193.

[F1] I. B. Frenkel, *Lectures on infinite dimensional Lie algebras*, Yale University, 1986.

[F2] I. B. Frenkel, *Kac-Moody algebras and dual resonance models*, in *Applications of Group Theory in Physics and Mathematical Physics*, eds. M. Flato et al, Lect. in Appl. Math. **21**, 325-353. AMS, Providence, 1985.

[FJ] I. B. Frenkel and N. Jing, *Vertex representations of quantum affine algebras*, Proc. Nat. Acad. Sci. USA (1988), 9373-9377.

[FK] I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980) 23–66.

[FJW1] I. B. Frenkel, N. Jing and W. Wang, *Vertex representations via finite groups and the McKay correspondence*, Internat. Math. Res. Notices **4** (2000), 195-222.

[FJW2] I. B. Frenkel, N. Jing and W. Wang, *Quantum vertex representations via finite groups and the McKay correspondence*, Commun. Math. Phys. **211** (2000), 365-393.

[FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, *An E_8-approach to F_1*, Contemp. Math. **45** (1985), 99-120.

[FLM2] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.

[HH] P. N. Hoffman and J. F. Humphreys, *Hopf algebras and projective representations of $G \wr S_n$ and $G \wr A_n$*, Can. J. Math. **38** (1986), 1380-1458.

[J1] N. Jing, *Vertex operators, symmetric functions and the spin group Γ_n*, J. Alg. **138** (1991), 340-398.

[J2] N. Jing, *New twisted quantum current algebras*, In: Proc. of Shanghai Intern. Conference on Representation Theory, June 1998, Higher Education, Beijing and Springer-Verlag, Hong Kong, 2000. (math.QA/9901066)

[Jo] T. Józefiak, *Characters of projective representations of symmetric groups*, Expo. Math. **7** (1989), 193-247.

[LW] J. Lepowsky and R. L. Wilson, *Construction of the affine Lie algebra $A_1^{(1)}$*, Commun. Math. Phys. **62** (1978), 43-53.

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.

[Mc] J. McKay, *Graphs, singularities and finite groups*, Proc. Sympos. Pure Math. **37**, Amer. Math. Soc, Providence, RI, 1980, pp. 183-186.

[MRY] R. Moody, S. Rao and T. Yokonuma, *Toroidal Lie algebras and vertex representations*, Geom. Dedica. **35** (1990), 283-307.

[Mo] A. O. Morris, *The spin representation of the symmetric group*, Proc. London Math. Soc. **12** (1962), 55-76.

[S] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, J. Reine Angew. Math. **139** (1911), 155-250.

[Se] G. Segal, *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. **80** (1981), 301-342.

[Sr] J.-P. Serre, *Représentation linéaires des groupes finis*, Troi. ed. corrigée, Herman, Paris, 1978.

[St] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), 87-134.

[Ws] D. B. Wales, *Some projective representations of S_n*, J. Algebra **61** (1979), 37-57.

[Wl] C. T. C. Wall, *Graded Brauer groups*, J. Reine Angew. Math. **213** (1964), 187–199.

[W] W. Wang, *Equivariant K-theory and wreath products*, MPI preprint # **86**, August 1998; *Equivariant K-theory, wreath products and Heisenberg algebra*, Duke Math. J. **103** (2000), 1-23.

[Z] A. Zelevinsky, *Representations of finite classical groups, A Hopf algebra approach.* Lecture Notes in Mathematics, **869**. Springer-Verlag, Berlin-New York, 1981.

Department of Mathematics, Yale University, New Haven, CT 06520

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205
E-mail address: jing@math.ncsu.edu

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205
E-mail address: wqwang@math.ncsu.edu